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Sums of commutators in non-commutative Banach function spaces

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Abstract

Let M be a II_∞-factor and denote by τ its normal faithful semi-finite trace. For any rearrangement invariant Köthe function space X on $[0,+\infty[$, let $X(M,\tau)$ be the associated non-commutative Banach function space. This paper is concerned with ideals in M of the form $I_X(M,\tau) = M \cap X(M,\tau)$ that are contained in $L^p(M,\tau)$ for some $p>0$. It is proved that an element T in $I_X(M,\tau)$ is a finite sum of commutators of the form $[A,B]$ with $A \in I_X(M,\tau)$ and $B \in M$ if and only if the function $t \to \frac{1}{t}\int_{|\lambda|>\lambda_t(T)} \lambda \, d\nu_T(\lambda)$ belongs to X, where ν_T is the Brown spectral measure of T and $t \to \lambda_t(T)$ is the non-increasing rearrangement of the function $\lambda \to |\lambda|$ with respect to ν_T. This extends to general Banach function spaces a result obtained by Kalton for quasi-Banach ideals of compact operators and implies that the Dixmier's trace of a quasi-nilpotent element in $L^{1,\infty}(M,\tau)$ is always zero.

Keywords: Von Neumann algebras; Commutators; Generalized s-numbers; Non-commutative Banach function spaces

1. Introduction and preliminaries

1.1. Introduction

Let J_1, J_2 be two ideals of compact operators in a separable Hilbert space H and denote by $[J_1, J_2]$ the linear span of the commutators $[A,B] = AB - BA$ where $A \in J_1$ and $B \in J_2$. It has been shown by Dykema et al. [9] that $[J_1, J_2]$ coincides with

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$[I, B(H)]$ where $I = J_1 J_2$ and $[I, B(H)]$ denotes the linear span of the commutators $[A, B] = AB - BA$ with $A \in I$ and $B \in B(H)$. Recently, a complete characterization of $[I, B(H)]$ for large classes of two-sided ideals I of compact operators has been achieved by Kalton [16]. For the Schatten class C_p $(p \geqslant 1)$, we have

$$[C_p, B(H)] = \begin{cases} C_p & \text{if } p > 1, \\ \left\{ T \in C_1 \;\middle|\; \sum_{n=1}^{+\infty} \frac{|\lambda_1(T) + \lambda_2(T) + \cdots + \lambda_n(T)|}{n} < \infty \right\} & \text{if } p = 1, \end{cases}$$

where $(\lambda_1(T), \lambda_2(T), \ldots)$ is a listing of the non-zero eigenvalues of T in such a way that $|\lambda_1(T)| \geqslant |\lambda_2(T)| \geqslant \cdots$, each eigenvalue being repeated according to its algebraic multiplicity. This result for $p > 1$ goes back to Pearcy and Topping [20] (see also [1]), and Kalton [15] gave the spectral characterization of $[C_1, B(H)]$. Recently, Dykema et al. [9] proved that a self-adjoint element T in an ideal I of compact operators belongs to $[I, B(H)]$ if and only if $diag(\sigma_1(T), \sigma_2(T), \ldots) \in I$, where $\sigma_n(T) = \frac{\lambda_1(T) + \lambda_2(T) + \cdots + \lambda_n(T)}{n}$. By using this result, Kalton [16] obtained a complete spectral characterization of $[I, B(H)]$ when I is a *geometrically stable* ideal of compact operators. As a corollary, Dykema and Kalton [10] proved that if τ is a trace on a geometrically stable ideal I of compact operators and $T \in I$, then $\tau(T)$ depends only on the eigenvalues of T and their algebraic multiplicities. This fact is not obvious when T is non-normal and may be viewed as a "renormalized" Lidskii's trace theorem (cf. [18]). It implies for instance, by using A. Connes Theorem 4 on the non-commutative residue (cf. [4]):

Proposition. *Let P be a classical pseudodifferential operator of order $-n$ on a closed compact manifold M of dimension n, acting on the sections of a vector bundle E. For any riemannian metric on M, we have*

$$\frac{1}{n(2\pi)^n} \int_M \left(\int_{S_x^*} \operatorname{tr}(\sigma_P(x, \xi)) \omega_\xi \right) dvol(x) = \lim_{N \to +\infty} \left(\frac{1}{\ln(N)} \sum_{k=1}^{N} \lambda_k(P) \right),$$

where $\sigma_P(x, \xi)$ is the principal symbol of P of order n, ω_ξ is the Leray form of the cosphere $S_x^* = \{\xi \in T_x^* M \mid ||\xi|| = 1\}$ and $(\lambda_1(P), \lambda_2(P), \ldots)$ is a listing of all non-zero eigenvalues of P acting on the L^2-sections over M of the bundle E.

The *Wodzicki residue* $Re\, s(P) = \frac{1}{n(2\pi)^n} \int_M (\int_{S_x^*} tr(\sigma_P(x, \xi)) \omega_\xi)\, dvol(x)$ is thus a spectral quantity, a fact that agrees with A. Connes point of view about locality in non-commutative geometry.

The aim of this paper if to extend the results of [9,16] to the non-commutative Banach spaces $X(M, \tau)$ associated with a rearrangement invariant Köthe function space X on $[0, +\infty[$. Here, M is a II_∞-factor with trace τ and we set by definition

$$X(M, \tau) = \{T \in \widetilde{M} \mid (t \to \mu_t(T)) \in X\},$$

where $\widetilde{M}$ is the algebra of all τ-measurable operators affiliated with M and $t \to \mu_t(T)$ is the non-increasing rearrangement of T with respect to τ. The space $X(M,\tau)$ is complete for the norm $\|T\|_{X(M,\tau)} = \|\mu_.(T)\|_X$ and $I_X(M,\tau) = X(M,\tau) \cap M$ is an ideal in M. Examples of such non-commutative Banach spaces are the L^p-spaces $L^p(M,\tau)$ $(1 \leqslant p < +\infty)$, the Lorentz spaces $L^p_\rho(M,\tau)$ $(1 \leqslant p < +\infty)$ associated with a weight $\rho :]0,+\infty[\to \mathbb{R}_+$ and the Marcinkiewicz spaces $M^p_\rho(M,\tau)$ $(1 \leqslant p < +\infty)$. The Dixmier's ideal $L^{1,\infty}(M,\tau)$, which plays a role in the local index theorem for elliptic operators along the leaves of a foliation (cf. [2]) is a particular example of Marcinkiewicz space. Our main result is the following:

Theorem. *Let X be a rearrangement invariant Köthe function space on $[0,+\infty[$ that is contained in $L^p([0,+\infty[,dt)$ for some $p>0$. Let M be a II_∞-factor acting on a separable Hilbert space H and denote by τ its normal faithful semi-finite trace. For any element T in $I_X(M,\tau)$, the following conditions are equivalent*:

(i) *T is a finite sum of commutators of the form $[A,B] = AB - BA$ with $A \in I_X(M,\tau)$ and $B \in M$*;
(ii) *the measurable function $t \in]0,+\infty[\to \frac{1}{t}\int_{|\lambda|>\lambda_t(T)} \lambda \, d\nu_T(\lambda) \in \mathbb{C}$ belongs to X*;
(iii) *there exists $T_0 \in I_X(M,\tau)$ such that $|\frac{1}{t}\int_{|\lambda|>\lambda_t(T)} \lambda \, d\nu_T(\lambda)| \leqslant \mu_t(T_0)$ for almost every $t>0$*;
(iv) *there exists $T_0 \in I_X(M,\tau)$, $T_0 \geqslant 0$, such that $|\Sigma_T(r)| \leqslant r N_{T_0}(r)$ for any $r>0$*;
(v) *there exists $T_0 \in I_X(M,\tau)$, $T_0 \geqslant 0$, such that $|\Sigma_T(r)| \leqslant r \ln(\frac{\Pi_{T_0}(r)}{r^{N_{T_0}(r)}})$ for any $r>0$.*

Moreover, if one of these conditions is satisfied, T is the sum of less than 14 *commutators of the form $[A,B] = AB - BA$ with $A \in I_X(M,\tau)$ and $B \in M$.*

Here, ν_T is Brown's τ-spectral measure of T (cf. [3]) and $t \to \lambda_t(T)$ is the non-increasing rearrangement of the function $\lambda \to |\lambda|$ with respect to ν_T. Thus, the quantity

$$\frac{1}{t}\int_{|\lambda|>\lambda_t(T)} \lambda \, d\nu_T(\lambda)$$

stands for the mean value of the "*t first spectral values*" of T, and the quantities

$$N_T(r) = \nu_T(\sigma_r(T)), \quad \Sigma_T(r) = \int_{\sigma_r(T)} \lambda \, d\nu_T(\lambda),$$

$$\Pi_T(r) = \exp\left(\int_{\sigma_r(T)} \ln(|\lambda|) \, d\nu_T(\lambda)\right)$$

stand, respectively, for the "*number*", the "*sum*" and the "*continuous product*" of the spectral values in $\sigma_r(T) = \{\lambda \in \sigma(T) \mid |\lambda| \geqslant r\}$.

As a corollary, we get the following result about the spectrality of any Dixmier's trace τ_ω on $L^{1,\infty}(M,\tau)$:

Proposition. *Let M be a II_∞ factor with normal faithful semi-finite trace τ and denote by τ_ω a Dixmier's trace on $L^{1,\infty}(M,\tau)$. For any $T\in L^{1,\infty}(M,\tau)$ such that 0 is isolated in $\sigma(T)$, we have $\tau_\omega(T)=0$. In particular, $\tau_\omega(T)=0$ for any element $T\in L^{1,\infty}(M,\tau)$ that is quasi-nilpotent or that has finite spectrum.*

This paper is organized as follows. After some necessary preliminaries on non-commutative Banach function spaces (Section 1), we give in Section 2 a spectral characterization of self-adjoint sums of commutators (Theorem 1). Section 3 introduces the numbers $N_T(r), \Sigma_T(r), \Pi_T(r)$ and proves some of their elementary properties. The main result of this section is Theorem 2, which provides the following spectral estimates:

$$|Re\,\Sigma_T(r)-\Sigma_{Re(T)}(r)|\leqslant C^{st} r \ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right),$$
$$|Im\,\Sigma_T(r)-\Sigma_{Im(T)}(r)|\leqslant C^{st} r \ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right).$$

These estimates imply that $T\in I_X(M,\tau)$ satisfies property (v) of the main theorem if and only if $Re(T)$ and $Im(T)$ satisfy this condition, a fact that allows reducing the proof to the self-adjoint case. Section 4 is devoted to the proof of the main theorem (Theorem 3) and of its consequence on the Dixmier's trace of a quasi-nilpotent element.

1.2. Type II_∞-factors

Let M be a von Neumann algebra acting on a separable Hilbert space H. We say that M is a factor if its center $Z(M)=M\cap M'$ reduces to the scalars. Factors have been classified by Murray and von Neumann into types I_n ($n=1,2,\dots,+\infty$), II_1, II_∞ and III. In this paper, we shall mainly be concerned with II_∞-factors. Recall that a type II_∞-factor M has a normal faithful semi-finite trace τ (unique up to a constant) such that $\{\tau(E)\,|\,E=E^2=E^*\in M\}=[0,+\infty]$. For more information on II_∞-factors, we refer to the Dixmier's book [6]. Note that we have:

Proposition 1. *Let M be a II_∞-factor acting on a separable Hilbert space, and τ a normal faithful semi-finite trace on M. There exists an increasing strongly continuous function $\lambda\to E_\lambda$ from $[0,+\infty[$ into the lattice of orthogonal projections in M satisfying the following conditions*:

(i) $E_0=0$ *and* $E_\lambda\to I$ *when* $\lambda\to+\infty$;
(ii) $\tau(E_\lambda)=\lambda$ *for any* $\lambda\geqslant 0$.

Proof. Note first that any II_1-factor N has an increasing strongly continuous family $(Q_\lambda)_{0\leqslant\lambda\leqslant 1}$ of orthogonal projections such that $\tau_0(Q_\lambda)=\lambda$ for $\lambda\in[0,1]$, where τ_0 denotes the normalized trace of N. Indeed, by using [6, Corollaire 3, p. 219], we can construct inductively an increasing family $Q_{k2^{-n}}$ of projections in N indexed by the dyadic numbers $k/2^n$ of $[0,1]$ $(n,k\in\mathbb{Z})$ such that $\tau_0(Q_{k2^{-n}})=k2^{-n}$. For any $\lambda\in[0,1]$, set $Q_\lambda=\mathrm{Sup}_{k2^{-n}\leqslant\lambda}\, Q_{k2^{-n}}$. We thus define an increasing family of projections in N such that $\tau_0(Q_\lambda)=\lambda$ and which is strongly continuous by Dixmier [6, Lemma 1, p. 270].

Let now M be a II_∞-factor with normal faithful trace τ. We may assume w.l.o.g. that $M=N\otimes B(H_0)$ and $\tau=\tau_0\otimes Tr$, where N is a II_1-factor acting on a separable Hilbert space with normalized trace τ_0, H_0 is a separable infinite dimensional Hilbert space, and Tr the usual trace on $B(H_0)$. Let $(e_1,e_2,\dots)$ be an orthonormal basis for H_0 and denote by Q_n $(n\geqslant 1)$ the orthogonal projection on $\mathbb{C}e_n$. The projections $P_n=I\otimes Q_n\in M$ satisfy $I=\sum_{n\geqslant 1}P_n$ and $\tau(P_n)=1$ for any $n\geqslant 1$. Set $P_0=0$. Since the reduced von Neumann algebra M_{P_n} is a II_1-factor for $n\geqslant 1$, there exists an increasing strongly continuous family $(Q^n_\lambda)_{0\leqslant\lambda\leqslant 1}$ of projections in M such that $Q^n_0=0$, $Q^n_1=P_n$ and $\tau(Q^n_\lambda)=\lambda$. Set, for any $\lambda\in[0,+\infty[$:

$$E_\lambda=\sum_{n\leqslant\lambda}P_n+Q^{[\lambda]+1}_{\lambda-[\lambda]}.$$

We thus define a strongly continuous function $\lambda\to E_\lambda$ from $[0,+\infty[$ into the lattice of orthogonal projections in M, which is obviously non-decreasing and satisfies conditions (i) and (ii). $\square$

1.3. Generalized s-numbers

Let M be a semi-finite von Neumann algebra acting on a separable Hilbert space H and τ a normal semi-finite trace on M. For any densely defined closed operator T in H, denote by $T=U|T|$ its polar decomposition. We say that T is *affiliated with* M if the partial isometry U and the spectral projections $E_\lambda=\mathbb{1}_{]-\infty,\lambda]}(|T|)$ $(\lambda\in\mathbb{R})$ belong to M. A densely defined closed operator T affiliated with M is said to be τ-*measurable* if there exists, for any $\varepsilon>0$, a projection $E\in M$ such that $E(H)$ is contained in $Dom(T)$ and $\tau(I-E)<+\infty$. The set of all τ-measurable operators will be denoted by $\widetilde{M}$; it is a *-algebra for the natural algebraic operations on closed operators (see for instance [24]). For any $T\in\widetilde{M}$, the *non-increasing rearrangement* of T with respect to the trace τ is the function $t\to\mu_t(T)$ defined on $\mathbb{R}_+$ by

$$\mu_t(T)=\mathrm{Inf}\{||TE||,\ E=E^2=E^*\in M \text{ and } \tau(I-E)\leqslant t\}.$$

This function may be viewed as the non-increasing rearrangement of the function $t\to t$ on $\sigma(|T|)-\{0\}$ with respect to the spectral measure $\nu_{|T|}$ of $|T|$ defined by $\nu_{|T|}(f)=\tau(f(|T|))$, where f is any positive measurable function on $\sigma(|T|)-\{0\}$.

For more information on non-increasing rearrangements of operators, we refer to [13]. An element $T \in M$ is called *τ-compact* if it is the norm limit of a sequence $(T_n)_{n \geqslant 1}$ of elements in M such that $\tau(Supp(T_n^*)) < +\infty$. The ideal of all *τ-compact* elements in M will be denoted by $K_\tau(M)$. By Fack [11, Proposition 1.9, p. 315], $T \in M$ is *τ-compact* if and only if $\mu_t(T) \xrightarrow[t \to +\infty]{} 0$. For any $T \in K_\tau(M)$ and $t > 0$ such that $\mu_t(T) > 0$, we have:

$$\tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|)) = mes\{s > 0 \mid \mu_s(T) > \mu_t(T)\} \leqslant t$$
$$\leqslant mes\{s > 0 \mid \mu_s(T) \geqslant \mu_t(T)\} = \tau(\mathbb{1}_{[\mu_t(T),+\infty[}(|T|)) < +\infty$$

by Fack and Kosaki [13, Remark 3.3, p. 280]. If $\mu_t(T) = 0$, we have

$$\tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|)) = \tau(\mathbb{1}_{]0,+\infty[}(|T|)) \leqslant t < +\infty$$

(cf. [13, Proposition 2.2, p. 274]).

1.4. Non-commutative Banach function spaces

Let X be a rearrangement invariant Köthe function space on $[0, +\infty[$ (see [19, p. 118] for the definition). For any infinite semi-finite von Neumann algebra M acting on a separable Hilbert space H and any normal faithful semi-finite trace τ on M, consider the non-commutative Banach function space $X(M,\tau) = \{T \in \widetilde{M} \mid (t \to \mu_t(T)) \in X\}$ with the norm $\|T\|_{X(M,\tau)} = \|\mu_.(T)\|_X$. The completeness of $X(M,\tau)$ follows from [7, Theorem 4.5, p. 596]. Since we have $\mu_t(ATB) \leqslant \|A\|\|B\|\mu_t(T)$ for any $t > 0$, $A, B \in M$ and $T \in \widetilde{M}$, the space $I_X(M,\tau) = X(M,\tau) \cap M$ is an ideal in M. Let us give examples of non-commutative Banach function spaces $X(M,\tau)$.

Example 1. The L_p-spaces $L_p(M,\tau)$ $(1 \leqslant p < +\infty)$.

Example 2. The non-commutative Lorentz spaces $L^p_\rho(M,\tau)$ $(1 \leqslant p < +\infty)$ associated with a positive non-increasing continuous Lorentz weight $\rho :]0, +\infty[\to \mathbb{R}_+$ satisfying the following conditions: $\lim_{t \to +\infty} \rho(t) = 0$, $\int_0^1 \rho(t)\, dt < +\infty$ and $\int_0^{+\infty} \rho(t)\, dt = +\infty$.

Denote by $M(\mathbb{R}_+)$ the space of all complex measurable functions on $\mathbb{R}_+$ and by f^* the non-increasing rearrangement of a positive $f \in M(\mathbb{R}_+)$. Let $X = L^p_\rho([0, +\infty[)$ be the rearrangement invariant Lorentz function space on $[0, +\infty[$ defined by

$$L^p_\rho([0, +\infty[) = \left\{ f \in M(\mathbb{R}_+) \;\middle|\; \int_0^{+\infty} |f|^*(t)^p \rho(t)\, dt < +\infty \right\},$$

and equipped with the norm $||f||_{p,\rho} = (\int_0^{+\infty} |f|^*(t)^p \rho(t)\,dt)^{1/p}$. We have

$$L^p_\rho(M,\tau) = \left\{ T \in \widetilde{M} \;\middle|\; \int_0^{+\infty} \mu_t(T)^p \rho(t)\,dt < +\infty \right\},$$
$$||T||_{p,\rho} = \left(\int_0^{+\infty} \mu_t(T)^p \rho(t)\,dt \right)^{1/p}.$$

For $\rho(t) = \frac{p}{q} t^{\frac{p}{q}-1}$ $(1 \leqslant p < q)$, we recover the non-commutative Banach function spaces $L^{q,p}(M,\tau)$ introduced by Kosaki [17]. For any $T \in L^p_\rho(M,\tau)$, we have (see for instance [21]) $\mu_t(T) \leqslant ||T||_{p,\rho} (\int_0^t \rho(s)\,ds)^{-\frac{1}{p}}$ for $t > 0$, so that $I_X(M,\tau) = X(M,\tau) \cap M$ (with $X = L^p_\rho([0,+\infty[))$ is contained in $L^r(M,\tau)$ if $(\int_0^t \rho(s)\,ds)^{-\frac{r}{p}}$ is integrable on $[1,+\infty[$. For the $L^{q,p}(M,\tau)$-spaces $(0 < p < q)$, this condition reduces to $r > q$. Note that the ideal $I_X(M,\tau)$ is always contained in $K_\tau(M)$ since we have $(\int_0^t \rho(s)\,ds)^{-\frac{1}{p}} \xrightarrow[t \to +\infty]{} 0$ by hypothesis.

Example 3. The non-commutative Marcinkiewicz spaces $M^p_\rho(M,\tau)$ $(1 \leqslant p < +\infty)$ associated with a Lorentz weight $\rho :]0,+\infty[\to \mathbb{R}_+$ as above.

These spaces are defined by

$$M^p_\rho(M,\tau) = \left\{ T \in \widetilde{M} \;\middle|\; \exists C > 0 \text{ such that } \int_0^t \mu_s(T)\,ds \leqslant C \left(\int_0^t \rho(s)\,ds \right)^{1/p} \right\};$$

the norm is given by $||T||^*_{p,\rho} = \operatorname{Sup}_{t>0} \left(\frac{\int_0^t \mu_s(T)\,ds}{(\int_0^t \rho(s)\,ds)^{1/p}} \right)$. Note that $M^1_\rho(M,\tau)$ is isometrically isomorphic to the dual of $L^1_\rho(M,\tau)$. For $\rho(t) = \frac{1}{t+1}$, the Marcinkiewicz space $M^1_\rho(M,\tau)$ is called the Dixmier's ideal and is denoted by $L^{1,\infty}(M,\tau)$; it is contained in M because the weight ρ is bounded. This non-commutative Banach function space appears naturally in the measured index theory of elliptic operators affiliated with von Neumann algebras such as uniformly elliptic almost periodic pseudo-differential operators on $\mathbb{R}^n$ (cf. [22]) or leafwise elliptic pseudo-differential operators on measured foliations (cf. [2]). For any $T \in M^p_\rho(M,\tau)$, we have $\mu_t(T) \leqslant ||T||^*_{p,\rho} \frac{1}{t} (\int_0^t \rho(s)\,ds)^{\frac{1}{p}}$ if $t > 0$, so that $I_X(M,\tau) = X(M,\tau) \cap M$ is contained in $L^r(M,\tau)$ if $\frac{1}{t^r} (\int_0^t \rho(s)\,ds)^{\frac{r}{p}}$ is integrable on $[1,+\infty[$. For the Dixmier's trace ideal $L^{1,\infty}(M,\tau)$, this condition is satisfied whenever $r > 1$. Note also that $I_X(M,\tau)$ is contained in $K_\tau(M)$ if $(\int_0^t \rho(s)\,ds)^{\frac{1}{p}} = o(t)$ when $t \to +\infty$. For more information on these non-commutative Banach spaces, we refer to [7] where the duality theory is studied. A detailed study of the non-commutative Lorentz and Marcinkiewicz spaces may also be found in [21].

Note finally that the non-commutative Banach function spaces $X(M,\tau)$ considered here are *geometrically stable*. More precisely, we have:

Proposition 2. *Let X be a rearrangement invariant Köthe function space on* $[0,+\infty[$ *and M an infinite semi-finite von Neumann algebra with a normal faithful semi-finite trace* τ. *For any* $T\in X(M,\tau)$, *the function* $t\in[0,+\infty[\to\Lambda_t(T)^{1/t}=\exp(\frac{1}{t}\int_0^t \ln(\mu_s(T))\,ds)\in\mathbb{R}_+$ *belongs to X.*

Proof. By Calderon's interpolation theorem, the operator D_s defined for any $s>0$ by $(D_s f)(t)=f(\frac{t}{s})$ maps continuously X into itself. Let us fix $r>0$ such that $r>\frac{\ln||D_2||}{\ln(2)}$. Since we have $||D_{2^k}||\leqslant||D_2||^k$ for any $k=0,1,\dots$, the serie $\sum_{k=0}^{\infty}2^{-rk}D_{2^k}(\mu_.(T))=f_T$ converges in X and defines a positive non-increasing function $f_T\in X$. Let $t>0$ be fixed. For any $s>0$ such that $t2^{-k-1}\leqslant s\leqslant t2^{-k}$, we have $\mu_s(T)\leqslant\mu_{t2^{-k-1}}(T)\leqslant 2^{r(k+1)}f_T(t)\leqslant(\frac{2t}{s})^r f_T(t)$, and hence $\mu_s(T)\leqslant(\frac{2t}{s})^r f_T(t)$ for any $s>0$ such that $s\leqslant t$. Integrating this inequality, we get

$$\exp\left(\frac{1}{t}\int_0^t \ln(\mu_s(T))\,ds\right)\leqslant e^{r(\ln(2)+1)}f_T(t)\quad\text{for any } t>0,$$

and the result follows. $\square$

1.5. Subharmonic functions

For the definition and elementary properties of subharmonic functions on the complex plane, we refer to [14]. Note that a subharmonic function that is not identically $-\infty$ is locally Lebesgue integrable, and the set of all points $z\in\mathbb{C}$ where $f(z)=+\infty$ has empty interior. For any subharmonic function $f:U\to[-\infty,+\infty]$ on a complex domain U that is not identically $-\infty$, there exists a unique Borel positive measure μ on U such that we have, for any compact subset K of U:

$$f(z)=\int_K \ln|z-u|\,d\mu(u)+h(z)\quad\text{if } z\in K,$$

where h is harmonic on $int(K)$ (see for instance [14, Theorem 3.9, p. 104]). Finally, note that the function $f(z)=\int_K f_x(z)\,d\mu(x)$ is subharmonic on a complex domain U for any family $(f_x)_{x\in K}$ of subharmonic functions on U indexed by a compact set K with a positive Radon measure μ such that:

(i) $(x,z)\to f_x(z)$ is measurable on $K\times\partial B(a,r)$ for all $\overline{B(a,r)}\subset U$;
(ii) $x\to f_x(z)$ is μ-integrable for all $z\in U$.

2. Spectral characterization of self-adjoint sums of commutators

2.1. Statement of the result

Let M be a II_∞-factor acting on a separable Hilbert space and denote by τ its unique (up to a positive constant) normal faithful semi-finite trace. Consider a rearrangement invariant Köthe function space X on $[0,+\infty[$ such that $|f|^*(t)\xrightarrow[t\to+\infty]{}0$ for any $f\in X$. Then, the elements of the ideal $I_X(M,\tau)=\{T\in M \mid \mu_.(T)\in X\}$ of M are τ-compact operators. In this section, we shall give (Theorem 1) a spectral characterization of the self-adjoint elements in $I_X(M,\tau)$ that are finite sums of commutators of the form $[A,B]=AB-BA$ with $A\in I_X(M,\tau)$ and $B\in M$. To this end, we need to define for any self-adjoint element $T\in K_\tau(M)$ a function $t\in]0,+\infty[\to\sigma_T(t)\in\mathbb{R}$ generalizing the sequence $(\frac{1}{n}\sum_{k=1}^n\lambda_k(T))_{n\geqslant1}$, where $(\lambda_n(T))_{n\geqslant1}$ is a listing of the non-zero eigenvalues of the compact operator $T\in B(H)$, counted according to algebraic multiplicity and arranged in such a way that $(|\lambda_n(T)|)_{n\geqslant1}$ is decreasing.

2.1.1. Definition of σ_T

For any $T\in K_\tau(M)$, let us call *interval of constancy* of the function $s\to\mu_s(T)$ any maximal interval where this function is constant. Since $s\to\mu_s(T)$ is continuous from the right, an interval of constancy is either of the form $[a,b]$ with $a<b$ (if $\mu_a(T)>0$ and $\mu(T)$ is continuous at b), or of the form $[a,b[$ with $a<b$ (if $\mu_a(T)>0$ and $\mu(T)$ is not continuous at b), or even of the form $[a,+\infty[$ (if $\mu_a(T)=0$).

Definition 1. Let M be an infinite semi-finite von Neumann algebra with a normal faithful semi-finite trace τ. For any self-adjoint τ-compact element $T\in M$ and any $t>0$, let $\sigma_T(t)$ be the real number defined as follows:

$$\sigma_T(t)=\frac{1}{t}\tau(T\mathbb{1}_{[\mu_t(T),+\infty[}(|T|))=\frac{1}{t}\tau(T\mathbb{1}_{]\mu_t(T),+\infty[}(|T|))$$

if $\mu_t(T)>0$ and t is not contained in an interval of constancy of $s\to\mu_s(T)$;

$$\sigma_T(t)=\frac{1}{t}\Big\{\tau(T\mathbb{1}_{]\mu_a(T),+\infty[}(|T|))+\frac{t-a}{b-a}\mu_t(T)[\tau(\mathbb{1}_{\{\mu_t(T)\}}(T)-\mathbb{1}_{\{-\mu_t(T)\}}(T))]\Big\}$$

if $\mu_t(T)>0$ and t belongs to an interval of constancy $[a,b[$ or $[a,b]$ $(a<b)$ of $s\to\mu_s(T)$, and

$$\sigma_T(t)=\frac{\tau(T)}{t}\quad\text{if }\ \mu_t(T)=0.$$

The quantity $\sigma_T(t)$ may be viewed as the "*mean value of the t first spectral values of T of largest modulus*". This definition requires interpretation. If $\mu_t(T)>0$ and t is not contained in an interval of constancy of $s\to\mu_s(T)$, we have $\tau(\mathbb{1}_{[\mu_t(T),+\infty[}(|T|))=\tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|))=t$ by Fack and Kosaki [13, Remark 3.3, p. 280]. It follows that the operator $T\mathbb{1}_{[\mu_t(T),+\infty[}(|T|)$ is trace-class, and $\sigma_T(t)$ is well defined. If $\mu_t(T)>0$ and t belongs to an interval $[a,b[$ or $[a,b]$ $(a<b<+\infty)$ of constancy of $s\to\mu_s(T)$, we have $\tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|))=a$ and $\tau(\mathbb{1}_{[\mu_a(T),+\infty[}(|T|))=b$, so that the operator $T\mathbb{1}_{]\mu_t(T),+\infty[}(|T|)$ and the spectral projections $\mathbb{1}_{\{\mu_a(T)\}}(T)$, $\mathbb{1}_{\{-\mu_a(T)\}}(T)$ are trace-class. It follows that $\sigma_T(t)$ is well defined in this case. Finally, if $\mu_t(T)=0$ we have $\mu_s(T)=0$ for $s\geqslant t$. In this case, T is trace-class and $\sigma_T(t)$ is again well defined. Note that we have $\sigma_T(s)=\frac{\tau(T)}{s}$ for $s\geqslant t$.

Remark. For any self-adjoint τ-compact element $T\in M$, we have:

$$\left|\frac{1}{t}\tau(T\mathbb{1}_{]\mu_t(T),+\infty[}(|T|))\right|\leqslant|\sigma_T(t)|+\mu_t(T)$$

and

$$|\sigma_T(t)|\leqslant\left|\frac{1}{t}\tau(T\mathbb{1}_{]\mu_t(T),+\infty[}(|T|))\right|+\mu_t(T).$$

These inequalities are obvious if $\mu_t(T)>0$ and t is not contained in an interval of constancy of $s\to\mu_s(T)$. If $\mu_t(T)>0$ and t belongs to an interval $[a,b[$ or $[a,b]$ $(a<b<+\infty)$ of constancy of $s\to\mu_s(T)$, we have

$$\begin{aligned}\left|\frac{t-a}{b-a}\mu_a(T)[\tau(\mathbb{1}_{\{\mu_a(T)\}}(T)-\mathbb{1}_{\{-\mu_a(T)\}}(T))]\right|&\leqslant\frac{t-a}{b-a}\mu_t(T)\tau(\mathbb{1}_{\{\mu_a(T)\}}(|T|))\\&\leqslant(t-a)\mu_t(T),\end{aligned}$$

and hence $|\frac{1}{t}\tau(T\mathbb{1}_{]\mu_t(T),+\infty[}(|T|))|\leqslant|\sigma_T(t)|+\frac{t-a}{t}\mu_t(T)\leqslant|\sigma_T(t)|+\mu_t(T)$. In the same way, we get $|\sigma_T(t)|\leqslant|\frac{1}{t}\tau(T\mathbb{1}_{]\mu_t(T),+\infty[}(|T|))|+\mu_t(T)$. Finally, if $\mu_t(T)=0$ we have

$$\sigma_T(t)=\frac{1}{t}\tau(T\mathbb{1}_{[0,+\infty[}(|T|))=\frac{1}{t}\tau(T\mathbb{1}_{]0,+\infty[}(|T|)),$$

and the above inequalities are true.

2.1.2. The main result

Theorem 1. *Let X be a rearrangement invariant Köthe function space on $[0,+\infty[$ such that $|f|^*(t)\xrightarrow[t\to+\infty]{}0$ for any $f\in X$. Let M be a II_∞-factor acting on a separable Hilbert space H and denote by τ its normal faithful semi-finite trace. For any self-adjoint element T in $I_X(M,\tau)$, the following conditions are equivalent*:

(i) *T is a finite sum of commutators of the form $[A,B]=AB-BA$ with $A\in I_X(M,\tau)$ and $B\in M$.*

(ii) *The measurable function* $t \in]0, +\infty[\to \sigma_T(t) \in \mathbb{R}$ *belongs to* X.
(iii) *The measurable function* $t \in]0, +\infty[\to \frac{1}{t}\tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|)T) \in \mathbb{R}$ *belongs to* X.

Moreover, if one of these conditions is satisfied, T *is the sum of less than 7 commutators of the form* $[A,B] = AB - BA$ *with* $A \in I_X(M,\tau)$ *and* $B \in M$.

The proof of (i) ⇒ (iii) ⇒ (ii) is straightforward and works in any semi-finite von Neumann algebra M. The proof of (ii) ⇒ (i) is based on a result of [12] describing the kernel of the trace in a finite factor.

2.2. *Proof of theorem 1*

2.2.1. *Proof of* (i) ⇒ (iii) ⇒ (ii)

(i) ⇒ (iii): Let T be a self-adjoint element in $I_X(M,\tau)$ and assume the existence of elements $A_1, A_2, \ldots, A_n$ in $I_X(M,\tau)$ and $B_1, B_2, \ldots, B_n$ in M such that $T = \sum_{i=1}^n [A_i, B_i]$. To prove (iii), it suffices to show that

$$\left|\frac{\tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|)T)}{t}\right| \leqslant 2n\mu_t(T) + 2(2n+1)\sum_{i=1}^n ||B_i||\mu_t(A_i) \quad \text{for any } t>0. \tag{2.1}$$

Let us set $\sigma'_T(t) = \frac{\tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|)T)}{t}$. By Fack and Kosaki [13, Proposition 2.2, p. 274], the spectral projection $E_t = \mathbb{1}_{[0,\mu_t(T)]}(|T|)$ satisfies $\tau(I - E_t) \leqslant t$. For $i = 1, 2, \ldots, n$, denote by E_t^i (resp. F_t^i) the spectral projection of $E_t|A_i|^2E_t$ (resp. of $E_t|A_i^*|^2E_t$) corresponding to the interval $[0, \mu_t^{E_t}(E_t|A_i|^2E_t)]$ (resp. $[0, \mu_t^{E_t}(E_t|A_i^*|^2E_t)]$), where $\mu_t^{E_t}(x)$ is the t-th generalized s-number associated with the reduced trace $X \to \tau(E_tXE_t)$ on E_tME_t. By Fack and Kosaki [13] again, the subprojections $E_t - E_t^i$ and $E_t - F_t^i$ of E_t satisfy $\tau(E_t - E_t^i) \leqslant t$ and $\tau(E_t - F_t^i) \leqslant t$, so that the projection $F_t = \mathrm{Sup}_{1 \leqslant i \leqslant n}\{E_t - E_t^i, E_t - F_t^i\}$ satisfies $\tau(F_t) \leqslant \sum_{i=1}^n \tau(E_t - E_t^i) + \tau(E_t - F_t^i) \leqslant 2nt$. Set $P_t = I - E_t + F_t$. From the relation $\tau(P_tT) = \tau(T\mathbb{1}_{]\mu_t(T),+\infty[}(|T|)) + \tau(F_tT)$, we get

$$|\sigma'_T(t)| \leqslant \left|\frac{\tau(F_tT)}{t}\right| + \left|\frac{\tau(P_tT)}{t}\right|. \tag{2.2}$$

Since $F_t = E_tF_t$, we have $|\tau(F_tT)| = |\tau(F_tTE_t)| \leqslant ||TE_t||\tau(F_t) = |||T|E_t||\tau(F_t) \leqslant \mu_t(T)\tau(F_t) \leqslant 2nt\mu_t(T)$, and hence

$$\left|\frac{\tau(F_tT)}{t}\right| \leqslant 2n\mu_t(T). \tag{2.3}$$

On the other hand, we have

$$P_tT = \sum_{i=1}^n P_t[A_i, B_i] = \sum_{i=1}^n ([P_tA_i, P_tB_i] + P_tA_i(I - P_t)B_i - P_tB_i(I - P_t)A_i)$$

and hence, by cyclicity of the trace τ:

$$\begin{aligned} |\tau(P_tT)| &= \left|\sum_{i=1}^{n} \tau(P_tA_i(I-P_t)B_i) - \tau(P_tB_i(I-P_t)A_i)\right| \\ &\leqslant \sum_{i=1}^{n} |\tau(P_tA_i(I-P_t)B_i)| + |\tau(P_tB_i(I-P_t)A_i)| \\ &\leqslant \sum_{i=1}^{n} \tau(P_t)||A_i(I-P_t)||||B_i|| + \tau(P_t)||B_i||||A_i^*(I-P_t)|| \\ &\leqslant (2n+1)t \sum_{i=1}^{n} ||B_i||(||A_i(I-P_t)|| + ||A_i^*(I-P_t)||). \end{aligned}$$

Since we have $I - P_t \leqslant E_t^i$, we get

$$\begin{aligned} ||A_i(I-P_t)||^2 &= ||(I-P_t)|A_i|^2(I-P_t)|| \leqslant ||E_t^i|A_i|^2E_t^i|| = ||E_t^iE_t|A_i|^2E_tE_t^i|| \\ &\leqslant \mu_t^{E_t}(E_t|A_i|^2E_t) \leqslant \mu_t(|A_i|^2) = \mu_t(A_i)^2, \end{aligned}$$

and hence $||A_i(I-P_t)|| \leqslant \mu_t(A_i)$ for $i = 1, 2, \ldots, n$. In the same way, we get

$$||A_i^*(I-P_t)|| \leqslant \mu_t(A_i^*) = \mu_t(A_i) \quad \text{for } i = 1, 2, \ldots, n,$$

so that finally

$$|\tau(P_tT)| \leqslant (2n+1)t \sum_{i=1}^{n} 2||B_i||\mu_t(A_i). \tag{2.4}$$

From (2.2)–(2.4), we get $|\sigma'_T(t)| \leqslant 2n\mu_t(T) + 2(2n+1)\sum_{i=1}^{n}||B_i||\mu_t(A_i)$, and (2.1) is proved.

(iii) ⇒ (ii): Follows from the remark after Definition 1. □

2.2.2. Proof of (ii) ⇒ (i)

The proof of (ii) ⇒ (i) is based on several lemmas.

Lemma 1. *Let M be a II_∞-factor acting on a separable Hilbert space H and denote by τ its normal faithful semi-finite trace. For any self-adjoint τ-compact element T in M, there exists an increasing sequence $(P_n)_{n\geqslant 0}$ of projections in M commuting with T such that*:

(i) $\tau(P_n) = 2^{n+1} - 1$ *for each* $n \geqslant 0$;

(ii) $\mathbb{1}_{]\mu_{2^{n+1}-1}(T),+\infty[}(|T|) \leqslant P_n \leqslant \mathbb{1}_{[\mu_{2^{n+1}-1}(T),+\infty[}(|T|)$ *for each* $n \geqslant 0$.

In particular, $\mathrm{Sup}_{n\geqslant 0}\, E_n \geqslant Supp(T)$.

Proof. *Step* 1: *Let us prove that there exists, for any* $t>0$, *a projection* $Q_t \in M$ *that commutes with* T *and satisfies* $\tau(Q_t) = t$; $\mathbb{1}_{]\mu_t(T),+\infty[}(|T|) \leqslant Q_t \leqslant \mathbb{1}_{[\mu_t(T),+\infty[}(|T|)$.

If $\mu_t(T)>0$, we have $\tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|)) \leqslant t \leqslant \tau(\mathbb{1}_{[\mu_t(T),+\infty[}(|T|))$ by τ-compactness of T (cf. [13, Remark 3.3, p. 280]). Since M is non-atomic, there exists a projection $Q'_t \in M$ such that $Q'_t \leqslant \mathbb{1}_{\{\mu_t(T)\}}(|T|)$ and $\tau(Q'_t) = t - \tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|))$. Set $Q_t = Q'_t + \mathbb{1}_{]\mu_t(T),+\infty[}(|T|)$; we thus define a projection in M that commutes with T and satisfies:

$$\tau(Q_t) = t; \quad \mathbb{1}_{]\mu_t(T),+\infty[}(|T|) \leqslant Q_t \leqslant \mathbb{1}_{[\mu_t(T),+\infty[}(|T|).$$

If $\mu_t(T) = 0$, we have $\tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|)) \leqslant t$ and $\tau(\mathbb{1}_{[\mu_t(T),+\infty[}(|T|)) = \tau(I) = +\infty$, so that:

$$\tau(\mathbb{1}_{\{\mu_t(T)\}}(|T|)) = +\infty.$$

Since M is non-atomic, we can choose a projection $Q'_t \in M$ satisfying $Q'_t \leqslant \mathbb{1}_{\{\mu_t(T)\}}(|T|)$ and $\tau(Q'_t) = t - \tau(\mathbb{1}_{]\mu_t(T),+\infty[}(|T|))$, and the conclusion follows as above.

Step 2: *Construction of the sequence* $(P_n)_{n\geqslant 0}$

Set $t_n = 2^{n+1} - 1$. Let us see by induction that it is possible to choose $P_n = Q_{t_n}$ so that $P_{n-1} \leqslant P_n$ for any $n \geqslant 1$. The resulting sequence $(P_n)_{n\geqslant 0}$ will have all the required properties by step 1. Assume that $P_0, P_1, \ldots, P_n$ have been constructed. If $\mu_{t_{n+1}}(T) < \mu_{t_n}(T)$, we have

$$P_n \leqslant \mathbb{1}_{[\mu_{t_n}(T),+\infty[}(|T|) \leqslant \mathbb{1}_{]\mu_{t_{n+1}}(T),+\infty[}(|T|) \leqslant Q_{t_{n+1}}$$

for any choice of $Q_{t_{n+1}}$, so that we may take $P_{n+1} = Q_{t_{n+1}}$ without any more precision. If $\mu_{t_{n+1}}(T) = \mu_{t_n}(T) > 0$, we may write by the induction hypothesis:

$$P_n = Q'_n + \mathbb{1}_{]\mu_{t_n}(T),+\infty[}(|T|),$$

where $Q'_n \leqslant \mathbb{1}_{\{\mu_{t_n}(T)\}}(|T|)$ is a projection in M such that $\tau(Q'_n) = t_n - \tau(\mathbb{1}_{]\mu_{t_n}(T),+\infty[}(|T|))$. Since we have

$$\begin{aligned}\tau(Q'_n) + \tau(\mathbb{1}_{]\mu_{t_n}(T),+\infty[}(|T|)) + t_{n+1} - t_n = t_{n+1} &\leqslant \tau(\mathbb{1}_{[\mu_{t_{n+1}}(T),+\infty[}(|T|)) \\ &= \tau(\mathbb{1}_{\{\mu_{t_n}(T)\}}(|T|)) + \tau(\mathbb{1}_{]\mu_{t_n}(T),+\infty[}(|T|)),\end{aligned}$$

we get $\tau(Q'_n) + t_{n+1} - t_n \leqslant \tau(\mathbb{1}_{\{\mu_{t_n}(T)\}}(|T|))$. But M is non-atomic, so that there exists a projection $Q'_{n+1} \in M$ with $Q'_n \leqslant Q'_{n+1} \leqslant \mathbb{1}_{\{\mu_{t_n}(T)\}}(|T|)$ such that

$$\tau(Q'_{n+1}) = \tau(Q'_n) + t_{n+1} - t_n = t_{n+1} - \tau(\mathbb{1}_{]\mu_{t_n}(T),+\infty[}(|T|)),$$

and the projection $P_{n+1} = Q'_{n+1} + \mathbb{1}_{]\mu_{t_{n+1}}(T),+\infty[}(|T|) \geqslant P_n$ satisfies all the required conditions. Finally, if $\mu_{t_{n+1}}(T) = \mu_{t_n}(T) = 0$, we have $\tau(\mathbb{1}_{\{\mu_{t_{n+1}}(T)\}}(|T|)) =$

$\tau(\mathbb{1}_{\{0\}}(|T|)) = +\infty$, and hence $\tau(\mathbb{1}_{\{0\}}(|T|) - Q'_n) = +\infty$. Since M is non-atomic, there exists a projection $Q'_{n+1} \in M$ such that

$$Q'_n \leqslant Q'_{n+1} \leqslant \mathbb{1}_{\{0\}}(|T|), \quad \tau(Q'_{n+1}) = t_{n+1} - \tau(\mathbb{1}_{]0,+\infty[}(|T|)),$$

and we can choose a projection $P_{n+1} \geqslant P_n$ satisfying all the required conditions. $\square$

Lemma 2. *Let X be a rearrangement invariant Köthe function space on $[0, +\infty[$ such that $|f|^*(t) \xrightarrow[t \to +\infty]{} 0$ for any $f \in X$. Let M be a II_∞-factor acting on a separable Hilbert space H and denote by τ its normal faithful semi-finite trace. Let T be a self-adjoint element in $I_X(M, \tau)$ such that the function $t \in]0, +\infty[\to \sigma_T(t) \in \mathbb{R}$ belongs to X, and denote by $(P_n)_{n\geqslant 0}$ an increasing sequence of projections in M satisfying the conditions of Lemma 1. Set $E_0 = P_0$, $E_n = P_n - P_{n-1}$ for $n \geqslant 1$, and $\alpha_n = \frac{\tau(E_n T)}{2^n}$ $(n \geqslant 0)$. Then, $T' = \sum_{n=0}^{+\infty} \alpha_n E_n \in M$ is a sum of two commutators of the form $[A, B] = AB - BA$ with $A \in I_X(M, \tau)$ and $B \in M$.*

Proof. Since $\tau(E_{n+1}) = 2\tau(E_n) = 2^{n+1}$ for $n \geqslant 0$ and M is II_∞-factor, we can write $E_{n+1} = E'_{n+1} + E''_{n+1}$ where E'_{n+1} and E''_{n+1} are two orthogonal projections in M such that $\tau(E'_{n+1}) = \tau(E''_{n+1}) = 2^n = \tau(E_n)$. By the comparability theorem (cf. [6, Proposition 13(iii), p. 236]), there exist partial isometries U'_n and U''_n in M such that:

$$\begin{cases} U'_n U'^*_n = E'_{n+1}, & U'^*_n U'_n = E_n, \\ U''_n U''^*_n = E''_{n+1}, & U''^*_n U''_n = E_n. \end{cases}$$

Set

$$\begin{cases} U_1 = \sum\limits_{n=0}^{\infty} s_n U'_n, & V_1 = \sum\limits_{n=0}^{\infty} U'^*_n, \\ U_2 = \sum\limits_{n=0}^{\infty} s_n U''_n, & V_2 = \sum\limits_{n=0}^{\infty} U''^*_n, \end{cases}$$

where $s_n = -\frac{\tau(P_n T)}{2^{n+1}}$ $(n \geqslant 0)$. Since $(s_n)_{n\geqslant 0}$ is bounded and the support and range projections of the U'_n's (resp. of the U''_n's) are pairwise disjoint, we thus define elements U_i, V_i in M $(i = 1, 2)$. By straightforward calculation, we get

$$[U_1, V_1] = \sum_{n=0}^{+\infty} s_n \left[U'_n, \sum_{k=0}^{\infty} U'^*_k \right] = \sum_{n=0}^{+\infty} s_n (U'_n U'^*_n - U'^*_n U'_n) = \sum_{n=0}^{+\infty} s_n (E'_{n+1} - E_n)$$

and, in the same way:

$$[U_2, V_2] = \sum_{n=0}^{+\infty} s_n (E''_{n+1} - E_n),$$

so that

$$[U_1, V_1] + [U_2, V_2] = \sum_{n=0}^{+\infty} s_n(E_{n+1} - 2E_n) = -2s_0E_0 + \sum_{n=1}^{+\infty} (s_{n-1} - 2s_n)E_n.$$

But $-2s_0 = \tau(P_0T) = \alpha_0$ and $s_{n-1} - 2s_n = \frac{\tau((P_n - P_{n-1})T)}{2^n} = \frac{\tau(E_nT)}{2^n} = \alpha_n$ for $n \geqslant 1$, so that we get

$$[U_1, V_1] + [U_2, V_2] = \sum_{n=0}^{+\infty} \alpha_n E_n = T'.$$

To end up the proof, let us show that $U_i \in I_X(M, \tau)$ for $i = 1, 2$. To this goal, note that we have

$$U_1^*U_1 = \sum_{n=0}^{+\infty} |s_n|^2 U_n'^* U_n' = \sum_{n=0}^{+\infty} |s_n|^2 E_n, \quad \text{and hence } |U_1| = \sum_{n=0}^{+\infty} |s_n| E_n.$$

Let $t > 0$ be fixed and consider the integer $n \geqslant 0$ such that $2^n - 1 < t \leqslant 2^{n+1} - 1$. Since $\tau(P_{n-1}) = 2^n - 1 < t$, we have $\mu_t(U_1) \leqslant |||U_1|(I - P_{n-1})|| = \mathrm{Sup}_{k \geqslant n}|s_k|$. For any $k \geqslant n$, the relation

$$\mathbb{1}_{]\mu_{2^{k+1}-1}(T), +\infty[}(|T|) \leqslant P_k \leqslant \mathbb{1}_{[\mu_{2^{k+1}-1}(T), +\infty[}(|T|)$$

implies that $P_kT = P_kT\mathbb{1}_{[\mu_{2^{k+1}-1}(T), \mu_t(T)]}(|T|) + T\mathbb{1}_{]\mu_t(T), +\infty[}(|T|)$ and hence

$$\begin{aligned}
|\tau(P_kT)| &\leqslant |\tau(P_k\mathbb{1}_{[\mu_{2^{k+1}-1}(T), \mu_t(T)]}(|T|)T)| + |\tau(\mathbb{1}_{]\mu_t(T), +\infty[}(|T|)T)| \\
&\leqslant ||\mathbb{1}_{[\mu_{2^{k+1}-1}(T), \mu_t(T)]}(|T|)T||\tau(P_k) + |\tau(\mathbb{1}_{]\mu_t(T), +\infty[}(|T|)T)| \\
&\leqslant \mu_t(T)\tau(P_k) + |\tau(\mathbb{1}_{]\mu_t(T), +\infty[}(|T|)T)| \\
&= \mu_t(T)(2^{k+1} - 1) + |\tau(\mathbb{1}_{]\mu_t(T), +\infty[}(|T|)T)|.
\end{aligned}$$

It follows that

$$\begin{aligned}
|s_k| = \frac{|\tau(P_kT)|}{2^{k+1}} &\leqslant \mu_t(T)\left(\frac{2^{k+1} - 1}{2^{k+1}}\right) + \frac{1}{2^{k+1}}|\tau(\mathbb{1}_{]\mu_t(T), +\infty[}(|T|)T)| \\
&\leqslant \mu_t(T) + \frac{1}{t}|\tau(\mathbb{1}_{]\mu_t(T), +\infty[}(|T|)T)| \leqslant \mu_t(T) + |\sigma_T(t)| + \mu_t(T) = 2\mu_t(T) + |\sigma_T(t)|,
\end{aligned}$$

a fact which implies that $\mu_t(U_1) \leqslant \mathrm{Sup}_{k \geqslant n}|s_k| \leqslant 2\mu_t(T) + |\sigma_T(t)|$ for any $t > 0$. Since the function $t \in]0, +\infty[\to \sigma_T(t) \in \mathbb{R}$ belongs to X, we get that $U_1 \in I_X(M, \tau)$. In the same way, we show that $U_2 \in I_X(M, \tau)$ and the proof is complete. $\square$

Lemma 3. *Let X be a rearrangement invariant Köthe function space on $[0,+\infty[$ such that $|f|^*(t)\xrightarrow[t\to+\infty]{}0$ for any $f\in X$. Let M be a II_∞-factor acting on a separable Hilbert space H and denote by τ its normal faithful semi-finite trace. Let T be a self-adjoint element in $I_X(M,\tau)$ such that the function $t\in]0,+\infty[\to\sigma_T(t)\in\mathbb{R}$ belongs to X, and denote by $(P_n)_{n\geqslant 0}$ an increasing sequence of projections in M satisfying the conditions of Lemma 1. Set $E_0=P_0,\ E_n=P_n-P_{n-1}$ for $n\geqslant 1$, and $\alpha_n=\frac{\tau(E_nT)}{2^n}$ $(n\geqslant 0)$. Then, $T''=\sum_{n=0}^{+\infty}E_n(T-\alpha_nI)E_n\in M$ is a sum of five commutators of the form $[A,B]=AB-BA$ with $A\in I_X(M,\tau)$ and $B\in M$.*

Proof. For each $n\geqslant 0$, set $T_n''=E_n(T-\alpha_nI)E_n\in E_nME_n$. Since

$$\alpha_n=\frac{\tau(E_nTE_n)}{2^n}=\frac{\tau(E_nTE_n)}{\tau(E_n)},$$

we get $\tau(T_n'')=0$. Moreover, we have

$$||E_nTE_n||=||E_n|T|E_n||=||(P_n-P_{n-1})|T|||\leqslant\mu_{2^n-1}(T)$$

and

$$\begin{aligned}||\alpha_nE_n||=|\alpha_n|&=\frac{|\tau(E_nT)|}{2^n}\leqslant\frac{\tau((P_n-P_{n-1})|T|(P_n-P_{n-1}))}{2^n}\leqslant\frac{\mu_{2^n-1}(T)\tau(E_n)}{2^n}\\&=\mu_{2^n-1}(T),\end{aligned}$$

so that $||T_n''||\leqslant 2\mu_{2^n-1}(T)\leqslant 2||T||$. Since E_nME_n is a finite factor, there exists by Fack de la Harpe [12, Lemme 2.2] elements A_n^i and B_n^i in E_nME_n $(1\leqslant i\leqslant 5)$ such that

$$||A_n^i||\leqslant 12||T_n''||\leqslant 24\mu_{2^n-1}(T)\leqslant 24||T||,\quad ||B_n^i||\leqslant 2\quad\text{and}\quad T_n''=\sum_{i=1}^{5}[A_n^i,B_n^i].$$

Set $A^i=\sum_{n=0}^{+\infty}A_n^i$ and $B^i=\sum_{n=0}^{+\infty}B_n^i$. Since the E_n's are pairwise orthogonal projections in M, we thus define bounded operators A_i,B_i in M $(i=1,2,\dots,5)$ which satisfy

$$T''=\sum_{n=0}^{+\infty}T_n''=\sum_{n=0}^{+\infty}\sum_{i=1}^{5}[A_n^i,B_n^i]=\sum_{i=1}^{5}[A^i,B^i].$$

To achieve the proof, it suffices to show that $A^i\in I_X(M,\tau)$ for $i=1,2,\dots,5$. To this end, fix $t\geqslant 1$ and consider the integer $n\geqslant 1$ such that $2^n-1\leqslant t<2^{n+1}-1$. Since $\tau(P_{n-1})=2^n-1\leqslant t$, we have:

$$\mu_t(A^i)\leqslant|||A^i|(I-P_{n-1})||=\operatorname*{Sup}_{k\geqslant n}||A_k^i||\leqslant 24\mu_{2^n-1}(T),$$

and we deduce from the obvious inequality $\frac{t}{4} < 2^{n-1} - \frac{1}{4} < 2^n - 1$ that $\mu_t(A^t) \leqslant 24\mu_{t/4}(T)$ if $t \geqslant 1$. It follows that $\mu_t(A^i) \leqslant 24(\mu_{t/4}(T) + ||T|| \mathbb{1}_{[0,1]}(t))$ for any $t > 0$, and since X is stable under the dilation operator D_4 and contains all the measurable compactly supported bounded functions, we get that $t \to \mu_t(A^i)$ belongs to X. Consequently, $A^i \in I_X(M, \tau)$ for $i = 1, 2, \ldots, 5$ and the proof of the lemma is complete. $\square$

End of the proof of (ii) $\Rightarrow$ (i): Let T be a self-adjoint element in $I_X(M, \tau)$ such that the function $t \in]0, +\infty[\to \sigma_T(t) \in \mathbb{R}$ belongs to X. By Lemma 1, there exists an increasing sequence $(P_n)_{n \geqslant 0}$ of projections in M commuting with T such that

$$\tau(P_n) = 2^{n+1} - 1 \quad \text{and} \quad \mathbb{1}_{]\mu_{2^{n+1}-1}(T), +\infty[}(|T|) \leqslant P_n \leqslant \mathbb{1}_{[\mu_{2^{n+1}-1}(T), +\infty[}(|T|).$$

Let $(E_n)_{n \geqslant 0}$ and $(\alpha_n)_{n \geqslant 0}$ be as in Lemma 2, and set $E_\infty = I - \sum_{n=0}^{+\infty} E_n$. Since $Supp(T) \leqslant \mathrm{Sup}_{n \geqslant 0} E_n$, we have $E_\infty T E_\infty = 0$ and hence $T = \sum_{n=0}^{+\infty} E_n T E_n = T' + T''$, where $T' = \sum_{n=0}^{+\infty} \alpha_n E_n$ and $T'' = \sum_{n=0}^{+\infty} E_n (T - \alpha_n I) E_n$. By Lemmas 2 and 3, we get that T is the sum of (at least) seven commutators of the form $[A, B] = AB - BA$ with $A \in I_X(M, \tau)$ and $B \in M$. $\square$

3. Some spectral inequalities

To extend Theorem 1 to non-self-adjoint elements T in $I_X(M, \tau)$, we shall assume that $T \in L^p(M, \tau)$ for some $p > 0$ in order to use the Brown's τ-spectral measure ν_T of T (see [3]). This τ-spectral measure allows to define for any $r > 0$ the "*number*" $N_T(r)$ (resp. the "*sum*" $\Sigma_T(r)$, the "*continuous product*" $\Pi_T(r)$) of all spectral values in $\sigma_r(T) = \{\lambda \in \sigma(T) \mid |\lambda| \geqslant r\}$. These spectral quantities will be used in the next section to extend Theorem 1 thanks to an estimate of $Re\Sigma_T(r) - \Sigma_{Re(T)}(r)$ which is the main result of this section. Let us first recall some facts about τ-spectral measures.

3.1. Preliminaries on spectral measures

3.1.1. Spectral measure of a compact operator

Let H be a separable Hilbert space and denote by $B(H)$ (resp. $K(H)$) the algebra of all bounded operators acting on H (resp. the ideal of all compact operators on H). Recall that the non-zero spectrum $\sigma^*(T) = \sigma(T) \backslash \{0\}$ of $T \in K(H)$ coincides with the set of all non-zero eigenvalues of T. The *spectral measure* of $T \in K(H)$ is the discrete measure ν_T on $\sigma^*(T)$ defined by $\nu_T(\{\lambda\}) = dim\, Ker(T - \lambda I)$ $(\lambda \in \sigma^*(T))$. We have (cf. [8]):

(i) $d\nu_{T^*}(\lambda) = d\nu_T(\bar{\lambda})$ for any $T \in K(H)$;
(ii) $\nu_{ST} = \nu_{TS}$ for any $T \in K(H)$ and $S \in B(H)$;

(iii) $f_*(\nu_T) = \nu_{f(T)}$ for any $T \in K(H)$ and any complex function f which is holomorphic in a neighborhood of $\sigma(T) \cup \{0\}$ and vanishes at 0.

3.1.2. Spectral measure of a trace-class operator

Let us denote by $C_1(H)$ the space of all trace-class operators acting on H. The spectral measure ν_T of $T \in C_1(H)$ is related to the function $F(z) = det(I - zT)$, where $det(I + T)$ denotes the *Fredholm determinant of* T defined by $det(I + T) = \sum_{n=0}^{\infty} Tr(\Lambda^n(T))$ (since T is trace-class, the serie converges absolutely by the H. Weyl inequalities). More precisely, the spectral measure ν_T of $T \in C_1(H)$ can be defined from the subharmonic function $z \to \ln det(|I - zT|)$ and the H. Weyl inequalities (resp. the Lidskii theorem on the trace) follow from the Poisson–Jensen formula (resp. from the Hadamard factorization of $det(I - zT)$). Consider indeed the entire function $F(z) = det(I - zT)$, which satisfies $\ln|F(z)| = o(|z|)$ when $|z| \to +\infty$ and whose zeros are the inverses of the non-zero eigenvalues λ of T (the order of $1/\lambda$ being exactly $\nu_T(\{\lambda\})$). Since $\sum_{\lambda \in \sigma^*(T)} |\lambda| \nu_T(\{\lambda\}) \leqslant Tr(|T|) < \infty$, we have

$$det(I - zT) = \prod_{\lambda \in \sigma^*(T)} (1 - z\lambda)^{\nu_T(\{\lambda\})} \tag{3.1}$$

thanks to Hadamard's factorization theorem (see for instance [23]). By identifying the coefficients of z in the Taylor expansion of each side of (3.1), we get the Lidskii theorem on the trace:

$$Tr(T) = \sum_{\lambda \in \sigma^*(T)} \lambda \nu_T(\{\lambda\}) \quad \left(= \int_{\sigma^*(T)} \lambda \, d\nu_T(\lambda) \right).$$

It also follows from (3.1) that $z \to \ln |det(I - zT)| = \ln det(|I - zT|)$ is a subharmonic function that satisfies (in the distribution sense)

$$\nabla^2(\ln det(|I - zT|)) = 2\pi \sum_{\lambda \in \sigma^*(T)} \nu_T(\{\lambda\}) \delta_{1/\lambda},$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. We thus have for any $T \in C_1(H)$,

$$d\nu_T(\lambda) = d\sigma_T(\lambda^{-1}),$$

where

$$d\sigma_T = \frac{1}{2\pi} \nabla^2(\ln det(|I - zT|)) = \frac{1}{2\pi} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\ln det(|I - zT|))$$

is the *Riesz measure* of the subharmonic function $z \to \ln det(|I - zT|)$. The H. Weyl inequalities can be recovered from the subharmonicity of $z \to \ln det(|I - zT|)$ as follows. Let $(\lambda_n(T))_{n \geqslant 1}$ be a listing of the non-zero eigenvalues of $T \in C_1(H)$

repeated according to algebraic multiplicity in such a way that $(|\lambda_n(T)|)_{n\geqslant 1}$ is non-increasing. From the Poisson–Jensen formula and the inequality $det(|I+T|)\leqslant det(I+|T|)$, we get

$$\sum_{|\lambda_n(T)|\geqslant 1/r} \ln(r|\lambda_n(T)|) = \frac{1}{2\pi}\int_0^{2\pi} \ln det(|I-re^{i\theta}T|)\,d\theta \leqslant \ln det(I+r|T|)$$
$$= \sum_{n=1}^{\infty} \ln(1+rc_n(T)),$$

where the $c_n(T)$ are the *characteristic numbers* of T. Applying this relation with T^k and r^k $(k\geqslant 1)$, we get

$$\sum_{|\lambda_n(T)|\geqslant 1/r} \ln(r|\lambda_n(T)|) = \frac{1}{k}\sum_{|\lambda_n(T)|\geqslant 1/r} \ln(r^k|\lambda_n(T)|^k) \leqslant \frac{1}{k}\sum_{n=1}^{\infty} \ln(1+r^k c_n(T)^k).$$

But we have $\ln(1+r^k c_n(T)^k)\leqslant \ln(2)+k\ln(rc_n(T))$ for all $n\geqslant 1$ such that $c_n(T)\geqslant 1/r$, and hence

$$\sum_{|\lambda_n(T)|\geqslant 1/r} \ln(r|\lambda_n(T)|) \leqslant \sum_{c_n(T)\geqslant 1/r} \ln(rc_n(T)) + \frac{1}{k}O(1).$$

The classical H. Weyl inequalities $\sum_{n=1}^{\infty}\ln_+(|\lambda_n(T)|)\leqslant\sum_{n=1}^{\infty}\ln_+(c_n(T))$ follow by letting $k\to+\infty$ and taking $r=1$.

3.1.3. Regularized determinants

For any p with $0<p<+\infty$, denote by $C_p(H)$ the Schatten p-class of all p-summable bounded operators on H. It turns out that the Fredholm determinant $det(I-T)$ can be regularized so that it extends from $C_1(H)$ to $C_p(H)$. Indeed, set for any $T\in C_p(H)$ and any integer $k\geqslant p$:

$$det_k(I-T) = det(R_k(T)),$$
$$\text{where } R_k(z) = (1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^{k-1}}{k-1}\right)\ (z\in\mathbb{C})$$

denotes the *k-th elementary Weierstrass factor*. Since we have $R_k(T)=I+T^kf(T)$ where f is an entire function, the operator $S_k=I-R_k(T)$ is trace-class and the *regularized determinant* $det_k(I-T)=det(I-S_k)$ makes sense. If $k=1$, we obviously get $det_1(I-T)=det(I-T)$. For any $T\in C_p(H)$ and $k\geqslant p$, we have (cf. [8, Chapter XI, Section 9, pp. 1106–1114]):

$$det_{k+1}(I-T) = \exp\left(\frac{Tr(T^k)}{k}\right)det_k(I-T),$$

and the zeros of the function $det_k(I - zT)$ are the inverses $1/\lambda$ of the non-zero eigenvalues λ of T, the order of $1/\lambda$ being exactly $\nu_T(\{\lambda\})$. We still have a factorization:

$$det_k(I - zT) = \prod_{\lambda \in \sigma^*(T)} \left[(1 - z\lambda)\exp\left(\lambda z + \frac{(\lambda z)^2}{2} + \cdots + \frac{(\lambda z)^{k-1}}{k-1} \right) \right]^{\nu_T(\{\lambda\})}, \qquad (3.2)$$

but the multiplicativity of the Fredholm determinant is lost. Since $|det(R_k(zT))| = det(|R_k(zT)|)$ for any $T \in C_p(H)$, it easily follows from (3.2) that $d\nu_T(\lambda) = d\sigma_T(\lambda^{-1})$, where

$$d\sigma_T = \frac{1}{2\pi} \nabla^2 (\ln det(|R_k(zT)|)) = \frac{1}{2\pi} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\ln det(|R_k(zT)|))$$

is the *Riesz measure* of the subharmonic function $z \to \ln(det(|R_k(zT)|))$ (k integer with $p \leqslant k$). Moreover, as in the case $p = 1$, we may recover the H. Weyl inequalities and the Lidskii theorem from the Poisson–Jensen formula and the Hadamard factorization of $det_k(I - zT)$.

3.1.4. The Brown spectral measure

Let again M be a von Neumann algebra with a normal semi-finite trace τ. For any $X \in I + L^p(M, \tau) \cap M$, we shall denote by $\Delta(X) = \exp(\tau(\ln|X|))$ the Fuglede–Kadison determinant of X (which is zero if $Ker\, X \neq \{0\}$ or if $\ln|X| \notin L^1(M, \tau)$). Brown [3, Theorem 3.3] showed that if $T \in L^p(M, \tau) \cap M$ $(0 < p < \infty)$, the function $z \to \ln \Delta(R_k(zT))$ is subharmonic on $\mathbb{C}$ for any integer $k \geqslant p$, a fact allowing defining the τ-*spectral measure* ν_T of T by

$$d\nu_T(\lambda) = d\sigma_T(\lambda^{-1}), \quad \text{where } d\sigma_T = \frac{1}{2\pi} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\ln \Delta(R_k(zT))).$$

This τ-spectral measure does not depend on the choice of the integer $k \geqslant p$ and is supported by $\sigma^*(T) = \sigma(T) \backslash \{0\}$. It is the unique non-negative measure ν on $\sigma^*(T)$ which satisfies

$$\ln \Delta(R_k(zT)) = \int_{\sigma^*(T)} \ln(|R_k(zu)|)\, d\nu(u),$$

and hence coincides for a normal $T \in L^p(M, \tau) \cap M$ with the usual spectral measure defined by $\nu_T(B) = \tau(\mathbb{1}_B(T))$ (where B is a Borel subset of $\sigma^*(T)$). The following theorem summarizes the main properties of Brown's τ-spectral measure:

Theorem (Brown [3]). *Let M be a von Neumann algebra with a normal faithful semi-finite trace τ. For any $T \in L^p(M, \tau) \cap M$, we have*:

(i) $\nu_{f(T)} = f_*(\nu_T)$ *for any function f which is holomorphic in a neighborhood of $\sigma(T)$ and vanishes at* 0;

(ii) $\int_0^t f(\lambda_s(T))\,ds \leqslant \int_0^t f(\mu_s(T))\,ds$ *for any* $t>0$ *and any non-decreasing function* $f : [0, +\infty[\to \mathbb{R}$ *such that* $f(e^t)$ *is convex and* $f(0)=0$, *where* $s \to \lambda_s(T)$ *is the non-increasing rearrangement of* $z \to |z|$ *with respect to* ν_T;

(iii) $\tau(f(T)) = \int_{\sigma^*(T)} f(\lambda)\,d\nu_T(\lambda)$ *and* $\ln \Delta(I+f(T)) = \int_{\sigma^*(T)} \ln(|1+f(\lambda)|)\,d\nu_T(\lambda)$ *for any function f which is holomorphic in a neighborhood of* $\sigma(T)$ *and vanishes to order at least* $k \geqslant p$ *at* 0 (*if* $0 \in \sigma(T)$);

(iv) $\nu_{TS} = \nu_{ST}$ *if TS and ST belong to* $T \in L^p(M,\tau) \cap M$.

3.2. Subharmonicity of $T \to \nu_T(f)$

Let again M be a semi-finite von Neumann algebra acting on a separable Hilbert space H, τ a normal faithful semi-finite trace on M and $T \in L^p(M,\tau) \cap M$ $(0<p<\infty)$. For any real continuous function f on $\mathbb{C}$ vanishing on a neighborhood of the origin, the number $\nu_T(f) = \int_{\sigma^*(T)} f(\lambda)\,d\nu_T(\lambda) \in \mathbb{C}$ is well defined and satisfies

$$|\nu_T(f)| \leqslant \left(\operatorname*{Sup}_{\lambda \in \sigma^*(T)} |f(\lambda)| \right) \nu_T(supp(f)).$$

The following result will be used in Section 3.4:

Proposition 3. *Let M be a semi-finite von Neumann algebra acting on a separable Hilbert space H,* τ *a normal faithful semi-finite trace on M and* $T, S \in L^p(M,\tau) \cap M$ $(0<p<+\infty)$. *For any continuous subharmonic function* $f : \mathbb{C} \to \mathbb{R}$ *vanishing on a neighborhood of the origin, the map* $z \in \mathbb{C} \to \nu_{T+zS}(f) = \int_{\sigma^*(T+zS)} f(\lambda)\,d\nu_{T+zS}(\lambda) \in \mathbb{R}$ *is subharmonic.*

Proof. Set $F(z) = \nu_{T+zS}(f)$ and $K(z) = \sigma^*(T+zS) \cap Supp(f)$. To prove the proposition, it suffices to show that F is subharmonic on any open disk $D(0,r) = \{z \in \mathbb{C} \mid |z| < r\}$. Fix $r>0$. Since f vanishes on a neighborhood of the origin, there exists two real numbers α and β with $0<\alpha<\beta$ such that:

(i) The interior of the corona $C_{\alpha,\beta} = \{\lambda \in \mathbb{C} \mid \alpha \leqslant |\lambda| \leqslant \beta\}$ contains $K(z)$ for any $z \in D(0,r)$;

(ii) $f(\lambda) = 0$ if $|\lambda| \leqslant \alpha$.

We thus may write, for $|z|<r$:

$$F(z) = \int_{C_{\alpha,\beta}} f(\lambda)\,d\nu_{T+zS}(\lambda).$$

By Poincare's "sweeping" trick (see for instance [5, Proposition 12.4, p. 362]), we know that there exists for any $R>\beta$ a subharmonic function g which coincides with f inside the disk $|\lambda| \leqslant \beta$ and such that $g(\lambda) = C^{st} + \ln(|\lambda|)$ for $|\lambda| \geqslant R$. Since $F(z)$ only depends, for $|z|<r$, on the restriction $f|C_{\alpha,\beta}$, we may assume w.l.o.g. that

f is harmonic outside the corona $C_{\alpha,2\beta} = \{\lambda \in \mathbb{C} \mid \alpha \leqslant |\lambda| \leqslant 2\beta\}$ and satisfies $max(f(\lambda), 0) = o(|\lambda|^n)$ as $|\lambda| \to +\infty$ for any integer $n \geqslant 1$. Let $d\sigma$ be the Riesz measure of f and set $K = \{u \in \mathbb{C} \mid \frac{1}{2\beta} \leqslant |u| \leqslant \frac{1}{\alpha}\}$. Since f is harmonic outside $C_{\alpha,2\beta}$, the measure $dv(u) = d\sigma(u^{-1})$ is supported by K and we can write (see for instance [3, Proposition 2.2]):

$$f(\lambda) = \int_K \ln(|R_k(\lambda u)|)\, dv(u), \tag{3.3}$$

where k is any fixed integer $\geqslant p$. But the integrand in (3.3) is $\leqslant C^{st} |\lambda u|^k$ thanks to the usual estimate of the k-th elementary Weierstrass factor, so that we get for any $z \in D(0, r)$ by Fubini's theorem:

$$\begin{aligned} F(z) &= \int_{C_{\alpha,\beta}} \left(\int_K \ln(|R_k(\lambda u)|)\, dv(u) \right) dv_{T+zS}(\lambda) \\ &= \int_K \left(\int_{C_{\alpha,\beta}} \ln(|R_k(\lambda u)|)\, dv_{T+zS}(\lambda) \right) dv(u) \\ &= \int_K \tau(\ln(|R_k(uT + zuS)|))\, dv(u) \\ &= \int_K \ln \Delta(R_k(uT + zuS))\, dv(u). \end{aligned}$$

Since the function $z \to \ln \Delta(R_k(uT + zuS))$ is subharmonic (cf. [3, Theorem 3.3]), the proposition follows. $\square$

3.3. *The numbers $N_T(r)$, $\Sigma_T(r)$ and $\Pi_T(r)$*

3.3.1. *Definition of the spectral numbers $N_T(r)$, $\Sigma_T(r)$, $\Pi_T(r)$*

Let M be a semi-finite von Neumann algebra acting on a separable Hilbert space H and τ a normal faithful semi-finite trace on M. For any $T \in L^p(M, \tau) \cap M$ $(0 < p < \infty)$ and $r > 0$, consider the set $\sigma_r(T) = \{\lambda \in \sigma(T) \mid |\lambda| \geqslant r\}$ of all spectral values of T of modulus $\geqslant r$ and define

$$N_T(r) = v_T(\sigma_r(T)), \quad \Sigma_T(r) = \int_{\sigma_r(T)} \lambda\, dv_T(\lambda),$$
$$\Pi_T(r) = \exp\left(\int_{\sigma_r(T)} \ln(|\lambda|)\, dv_T(\lambda) \right),$$

where v_T is the τ-spectral measure of T (cf. 3.1.4). The spectral quantity $N_T(r)$ (resp. $\Sigma_T(r), \Pi_T(r)$) stands for the "*number*" (resp. the "*sum*", the "*continuous product*") of all spectral values in $\sigma_r(T)$. If T is normal, we have $N_T(r) = \tau(\mathbb{1}_{[r,+\infty[}(|T|))$ and $\Sigma_T(r) = \tau(T\mathbb{1}_{[r,+\infty[}(|T|))$.

3.3.2. Elementary properties of $N_T(r)$, $\Sigma_T(r)$, $\Pi_T(r)$

Let us collect some properties of the spectral quantities $N_T(r), \Sigma_T(r)$, and $\Pi_T(r)$.

Lemma 4. *Let M be a semi-finite von Neumann algebra acting on a separable Hilbert space H and τ a normal faithful semi-finite trace on M. Let $T, S \in L^p(M,\tau) \cap M$ where $0 < p < +\infty$. For any $r > 0$, we have*:

(i) $N_T(r) = N_{r^{-1}T}(1)$;
(ii) $0 \leqslant N_T(r) = N_{T^*}(r) = N_{|T|}(r)$ *if T is normal*;
(iii) $N_{|T+S|}(r) \leqslant \lambda^{-1} N_{|\lambda^{-1}T|}(r) + (1-\lambda)^{-1} N_{|(1-\lambda)^{-1}S|}(r)$ *for any λ such that $0 < \lambda < 1$*;
(iv) $N_{Re(T)}(r) \leqslant 4N_{|T|}(r)$ *and* $N_{Im(T)}(r) \leqslant 4N_{|T|}(r)$.

Proof. (i) For any $r > 0$, we get by using [3, Theorem 4.1]:

$$N_T(r) = \int_{\sigma(T)} \mathbb{1}_{\{|\lambda| \geqslant r\}}(\lambda)\, d\nu_T(\lambda) = \int_{\sigma(T)} \mathbb{1}_{\{|\lambda| \geqslant 1\}}(r^{-1}\lambda)\, d\nu_T(\lambda)$$
$$= \int_{\sigma(r^{-1}T)} \mathbb{1}_{\{|\lambda| \geqslant 1\}}(\lambda')\, d\nu_{r^{-1}T}(\lambda') = N_{r^{-1}T}(1).$$

(ii) If T is normal, we have $N_T(r) = \tau(\mathbb{1}_{[r,+\infty[}(|T|)) = N_{|T|}(r) \geqslant 0$. Denote by m the Lebesgue measure on $\mathbb{R}$. By Fack and Kosaki [13, Remark 3.3, p. 280, and Lemma 2.5, p. 276], we get

$$N_T(r) = \tau(\mathbb{1}_{[r,+\infty[}(|T|)) = m(\{t \geqslant 0 \mid \mu_t(T) \geqslant r\}) = m(\{t \geqslant 0 \mid \mu_t(T^*) \geqslant r\}) = N_{T^*}(r),$$

and (ii) is proved.

(iii) Since we have $\mu_t(T+S) \leqslant \mu_{\lambda t}(T) + \mu_{(1-\lambda)t}(T)$ for any $\lambda \in]0,1[$ [13, Lemma 2.5, p. 276], the set $A = \{t \geqslant 0 \mid \mu_t(T+S) \geqslant r\}$ is contained for any $r > 0$ into the union:

$$\{\lambda^{-1}t \mid t \geqslant 0 \text{ and } \mu_t(T) \geqslant \lambda r\} \cup \{(1-\lambda)^{-1}t \mid t \geqslant 0 \text{ and } \mu_t(S) \geqslant (1-\lambda)r\}.$$

It follows that

$$\begin{aligned} N_{|T+S|}(r) &= m(A) \leqslant m(\{\lambda^{-1}t \mid t \geqslant 0 \text{ and } \mu_t(T) \geqslant \lambda r\}) \\ &\quad + m(\{(1-\lambda)^{-1}t \mid t \geqslant 0 \text{ and } \mu_t(S) \geqslant (1-\lambda)r\}) \\ &= m(\{\lambda^{-1}t \mid t \geqslant 0 \text{ and } \mu_t(\lambda^{-1}T) \geqslant r\}) \\ &\quad + m(\{(1-\lambda)^{-1}t \mid t \geqslant 0 \text{ and } \mu_t((1-\lambda)^{-1}S) \geqslant r\}) \\ &= \lambda^{-1} N_{|\lambda^{-1}T|}(r) + (1-\lambda)^{-1} N_{|(1-\lambda)^{-1}S|}(r). \end{aligned}$$

(iv) Since $Re(T)$ is self-adjoint, we get by (ii):

$$N_{Re(T)}(r) = N_{|\frac{T+T^*}{2}|}(r) \quad \text{for any } r>0,$$

and the first inequality follows from (iii) with $\lambda = 1/2$. We deduce the second inequality by changing T into $-iT$. $\square$

Lemma 5. *Let M be a semi-finite von Neumann algebra acting on a separable Hilbert space H and τ a normal faithful semi-finite trace on M. For any $T, S \in L^p(M,\tau) \cap M$ $(0<p<+\infty)$ and $r>0$, we have*:

(i) $\Sigma_T(r) = r\Sigma_{r^{-1}T}(1)$;
(ii) $|z\Sigma_T(r) - \Sigma_{zT}(r)| \leqslant rN_T(r)$ *for any $z \in \mathbb{C}$ such that $|z| \leqslant 1$*;
(iii) $|\Sigma_{Re(T)}(r) - Re\,\Sigma_T(r)| \leqslant rN_T(r)$ *if T is normal*;
(iv) $|\Sigma_{T_1}(r) + \cdots + \Sigma_{T_n}(r)| \leqslant (n-1)r[N_{T_1}(r) + \cdots + N_{T_n}(r)]$ *for any integer $n \geqslant 1$ and any finite sequence $T_1, T_2, \ldots, T_n$ of normal elements in $L^p(M,\tau) \cap M$ such that $T_1 + T_2 + \cdots + T_n = 0$.*

Proof. (i) By Brown [3, Theorem 4.1], we get

$$\Sigma_T(r) = \int_{\sigma^*(T)} \lambda \mathbb{1}_{\{|\lambda| \geqslant r\}}(\lambda)\, d\nu_T(\lambda) = r\int_{\sigma^*(r^{-1}T)} \lambda \mathbb{1}_{\{|\lambda| \geqslant 1\}}(\lambda)\, d\nu_{r^{-1}T}(\lambda) = r\Sigma_{r^{-1}T}(1).$$

(ii) We may assume w.l.o.g. that $z \neq 0$. For any $r>0$, we have

$$\begin{aligned}
|z\Sigma_T(r) - \Sigma_{zT}(r)| &= \left| z\int_{|\lambda| \geqslant r} \lambda\, d\nu_T(\lambda) - \int_{|\lambda| \geqslant r} \lambda\, d\nu_{zT}(\lambda) \right| \\
&= \left| z\int_{|\lambda| \geqslant r} \lambda\, d\nu_T(\lambda) - z\int_{|z\lambda| \geqslant r} \lambda\, d\nu_T(\lambda) \right| \\
&= |z| \left| \int_{r \leqslant |\lambda| < r|z|^{-1}} \lambda\, d\nu_T(\lambda) \right| \leqslant r\int_{r \leqslant |\lambda|} d\nu_T(\lambda) = rN_T(r).
\end{aligned}$$

(iii) If T is normal, we have

$$\begin{aligned}
|\Sigma_{Re(T)}(r) - Re\,\Sigma_T(r)| &= |\tau(Re(T)\mathbb{1}_{[r,+\infty[}(|Re(T)|)) - Re\,\tau(T\mathbb{1}_{[r,+\infty[}(|T|))| \\
&= |\tau(Re(T)\mathbb{1}_{\{|Re(\lambda)| \geqslant r\}}(T)) - \tau(Re\,T\mathbb{1}_{\{|\lambda| \geqslant r\}}(T))| \\
&= |\tau(Re(T)\mathbb{1}_{\{|Re(\lambda)| < r \leqslant |\lambda|\}}(T))| \leqslant r\tau(\mathbb{1}_{\{r \leqslant |\lambda|\}}(T)) = rN_T(r),
\end{aligned}$$

and (iii) is proved.

(iv) We may assume w.l.o.g. that $n>1$. Since each T_i is normal, we have $N_{T_i}(r) = \tau(E_i)$ and $\Sigma_{T_i}(r) = \tau(T_iE_i)$, where $E_i = \mathbb{1}_{\{|\lambda| \geqslant r\}}(T_i)$. Let us set

$E = E_1 \vee E_2 \vee \cdots \vee E_n$; we get

$$\left|\sum_{i=1}^{n} \Sigma_{T_i}(r)\right| = \left|\sum_{i=1}^{n} \tau(E_i T_i)\right| = \left|\sum_{i=1}^{n} \tau(ET_iE) - \tau((E-E_i)T_i(E-E_i))\right|$$

$$= \left|\tau\left(E\left(\sum_{i=1}^{n} T_i\right)E\right) - \sum_{i=1}^{n} \tau((E-E_i)T_i(E-E_i))\right|$$

$$= \left|\sum_{i=1}^{n} \tau((E-E_i)T_i(E-E_i))\right|.$$

Since we have $||(E-E_i)T_i(E-E_i)|| \leqslant r$ for $i = 1, 2, \ldots, n$, we finally get

$$\left|\sum_{i=1}^{n} \Sigma_{T_i}(r)\right| \leqslant r\sum_{i=1}^{n} \tau(E-E_i) = r\left(n\tau(E) - \sum_{i=1}^{n} \tau(E_i)\right) \leqslant (n-1)r\sum_{i=1}^{n} \tau(E_i)$$

$$= (n-1)r\sum_{i=1}^{n} N_{T_i}(r). \qquad \square$$

Lemma 6. *Let M be a semi-finite von Neumann algebra acting on a separable Hilbert space H and τ a normal faithful semi-finite trace on M. For any $T \in L^p(M,\tau) \cap M$ $(0 < p < +\infty)$ and $r > 0$, we have*

(i) $\Pi_T(r) = r^{N_T(r)} \Pi_{r^{-1}T}(1)$;
(ii) $1 \leqslant \frac{\Pi_T(r)}{r^{N_T(r)}} \leqslant \frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}} \leqslant \frac{\Pi_{\rho|T|}(r)}{r^{N_{\rho|T|}(r)}}$ *for any* $\rho \geqslant 1$;
(iii) $\rho^{N_T(r)} \leqslant \frac{\Pi_{\rho|T|}(r)}{r^{N_{\rho|T|}(r)}}$ *for any* $\rho \geqslant 1$.

Proof. (i) Follows immediately from [3, Theorem 4.1].

(ii) For any $r > 0$, we have $\ln \Pi_T(r) - \ln(r)N_T(r) = \int_{|\lambda| \geqslant 1} \ln(|\lambda|)\, d\nu_{r^{-1}T}(\lambda) \geqslant 0$ by (i), and hence $1 \leqslant \frac{\Pi_T(r)}{r^{N_T(r)}}$. Let $t \to \lambda_t(r^{-1}T)$ be the non-increasing rearrangement of the positive function $f(\lambda) = |\lambda|$ with respect to the measure $\nu_{r^{-1}T}$ on $\sigma^*(r^{-1}T)$. Since $t \to \ln_+(t)$ is continuous and vanishes at the origin, we have $\int_{|\lambda| \geqslant 1} \ln(|\lambda|)\, d\nu_{r^{-1}T}(\lambda) = \int_0^{+\infty} \ln_+(\lambda_t(r^{-1}T))\, dt$. It follows from [3, Theorem 3.6] that we have

$$\ln(\Pi_T(r)) - \ln(r)N_T(r) = \int_0^{+\infty} \ln_+(\lambda_t(r^{-1}T))\, dt \leqslant \int_0^{+\infty} \ln_+(\mu_t(r^{-1}T))\, dt$$
$$= \ln(\Pi_{|T|}(r)) - \ln(r)N_{|T|}(r),$$

and hence $\frac{\Pi_T(r)}{r^{N_T(r)}} \leqslant \frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}}$. To prove that $\frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}} \leqslant \frac{\Pi_{\rho|T|}(r)}{r^{N_{\rho|T|}(r)}}$ for any $\rho \geqslant 1$, we may assume by (i) that $r = 1$. In this case, we have

$$\ln(\Pi_{|T|}(1)) = \int_{|\lambda| \geqslant 1} \ln(|\lambda|)\, d\nu_{|T|}(\lambda) \leqslant \int_{|\rho\lambda| \geqslant \rho} \ln(|\rho\lambda|)\, d\nu_{|T|}(\lambda)$$

$$= \int_{|\lambda| \geqslant \rho} \ln(|\lambda|)\, d\nu_{\rho|T|}(\lambda) \leqslant \int_{|\lambda| \geqslant 1} \ln(|\lambda|)\, d\nu_{\rho|T|}(\lambda) = \ln(\Pi_{\rho|T|}(1)),$$

and the result follows.

(iii) To prove (iii), we may and do assume by (i) and Lemma 4(i) that $r = 1$. But we have

$$\ln(\rho) N_T(1) = \int_{|\lambda| \geqslant 1} \ln(\rho)\, d\nu_T(\lambda) \leqslant \int_{|\rho\lambda| \geqslant \rho} \ln(|\rho\lambda|)\, d\nu_T(\lambda)$$

$$= \int_{|\lambda| \geqslant \rho} \ln(|\lambda|)\, d\nu_{\rho T}(\lambda) \leqslant \int_{|\lambda| \geqslant 1} \ln(|\lambda|)\, d\nu_{\rho T}(\lambda)$$

$$= \ln(\Pi_{\rho T}(1)) \leqslant \ln(\Pi_{\rho|T|}(1)),$$

where the last inequality follows from assertion (ii). This proves (iii). $\square$

3.4. *A relation between* $Re\,\Sigma_T(r)$ *and* $\Sigma_{Re(T)}(r)$

3.4.1. *Regularization of* $\Sigma_T(r)$

Let $T \in L^p(M,\tau) \cap M$ $(0 < p < +\infty)$ and fix $r > 0$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function $0 \leqslant \varphi \leqslant 1$, such that $\varphi(t) = 0$ if $t \leqslant 0$ and $\varphi(t) = 1$ if $t \geqslant 1$. For any $r > 0$, consider the function $\psi_r : \mathbb{R} \to \mathbb{R}$ defined by $\psi_r(t) = \varphi(\ln(r^{-1}|t|))$ for any $t \in \mathbb{R}$. We thus define a C^∞ function $\psi_r(t)$ which is equal to 0 if $|t| \leqslant r$ and to 1 if $|t| \geqslant re$. The regularization $\Sigma_T^{reg}(r)$ of $\Sigma_T(r)$ is defined by

$$\Sigma_T^{reg}(r) = \int_{\sigma(T)} \lambda \psi_r(|\lambda|)\, d\nu_T(\lambda) \quad (r > 0),$$

where ψ_r is the above smooth approximation of the *cut-off* function $\mathbb{1}_{[r,+\infty[}$.

Lemma 7. *Let M be a semi-finite von Neumann algebra acting on a separable Hilbert space H and* τ *be a normal faithful semi-finite trace on M.*

(i) *For any* $r > 0$ *and any* $T \in L^p(M,\tau) \cap M$ $(0 < p < +\infty)$, *we have*

$$|\Sigma_T(r) - \Sigma_T^{reg}(r)| \leqslant erN_T(r);$$

(ii) *There exists a superharmonic function* $h:\mathbb{C}\to\mathbb{R}$ *and a constant* $C\geqslant 0$ *such that*:

$$\left| Re\,\Sigma_T^{reg}(r) - r\int_{\sigma^*(T)} h(r^{-1}\lambda)\,dv_T(\lambda)\right| \leqslant Cr\ln\left(\frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}}\right)$$

for any $r>0$ *and any* $T\in L^p(M,\tau)\cap M$ $(0<p<+\infty)$.

Proof. (i) Let $T\in L^p(M,\tau)\cap M$. For any $r>0$, we have

$$|\Sigma_T(r) - \Sigma_T^{reg}(r)| = \left|\int_{r\leqslant|\lambda|<er} \lambda(1-\psi_r(|\lambda|))\,dv_T(\lambda)\right|$$
$$\leqslant er\int_{r\leqslant|\lambda|} dv_T(\lambda) = erN_T(r).$$

(ii) Let $f:\mathbb{R}\to\mathbb{R}$ be the unique C^1 function which satisfies the differential equation $f''(t) = e^t(|\varphi''(t)| + 2|\varphi'(t)|)$ for $t\geqslant 0$ and vanishes for $t\leqslant 0$. This function is convex increasing and linear for $t\geqslant 1$; therefore, there exists a constant $C\geqslant 0$ such that

$$|f(t)|\leqslant C\max(t,0) \quad \text{for any } t\in\mathbb{R}.$$

A direct computation shows that the C^1 function $h:\mathbb{C}\to\mathbb{R}$ defined by

$$h(z) = -f(\ln|z|) + Re(z)\varphi(\ln|z|) \quad \text{if } z\neq 0, \text{ and } f(0)=0$$

satisfies $\nabla^2 h\leqslant 0$ and hence is superharmonic. Since we have for any $\lambda\in\mathbb{C}$:

$$r^{-1}Re(\lambda)\psi_r(|\lambda|) - h(r^{-1}\lambda) = f(\ln(r^{-1}|\lambda|)) \quad\text{and}\quad |f(\ln(r^{-1}|\lambda|))|\leqslant C\ln_+(r^{-1}|\lambda|),$$

we get

$$\left|Re\,\Sigma_T^{reg}(r) - r\int_{\sigma^*(T)} h(r^{-1}\lambda)\,dv_T(\lambda)\right| = \left|\int_{\sigma^*(T)} (Re(\lambda)\psi_r(|\lambda|) - rh(r^{-1}\lambda))\,dv_T(\lambda)\right|$$
$$\leqslant Cr\int_{r\leqslant|\lambda|} \ln(r^{-1}|\lambda|)\,dv_T(\lambda) = Cr\ln\left(\frac{\Pi_T(r)}{r^{N_T(r)}}\right),$$

and the conclusion follows from Lemma 6(ii). $\square$

3.4.2. An estimate for $|Re\,\Sigma_T(r) - \Sigma_{Re(T)}(r)|$

The following theorem is the main result of Section 3:

Theorem 2. *Let M be a semi-finite von Neumann algebra acting on a separable Hilbert space H and* τ *be a normal faithful semi-finite trace on M. Let* $T\in L^p(M,\tau)\cap M$ *where*

$0 < p < +\infty$. *There exists a constant* $C \geqslant 0$ *such that we have, for any* $r > 0$:

$$|Re\, \Sigma_T(r) - \Sigma_{Re(T)}(r)| \leqslant Cr \ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right) \text{ and } |Im\, \Sigma_T(r) - \Sigma_{Im(T)}(r)| \leqslant Cr \ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right).$$

Proof. *Step* 1: *Let us show that we have, for any* $r > 0$:

$$\Sigma_{Re(T)}(r) \leqslant 2\, Re\left(\int_0^{2\pi} \Sigma^{reg}_{T(\theta)}(r) \frac{d\theta}{2\pi}\right) + 184 r N_{|T|}(r),$$

where $T(\theta) = \frac{1}{2}(T + e^{i\theta} T^*) = \frac{1+e^{i\theta}}{2} Re(T) + i \frac{1-e^{i\theta}}{2} Im(T)$ $(0 \leqslant \theta \leqslant 2\pi)$.

Set $T_1 = T(\theta)$, $T_2 = -\frac{1+e^{i\theta}}{2} Re(T)$, and $T_3 = -i\frac{1-e^{i\theta}}{2} Im(T)$. We thus define normal operators in $T \in L^p(M, \tau) \cap M$ satisfying by Lemma 4(ii)–(iv):

$$N_{T_1}(r) = N_{|T_1|}(r) \leqslant 4 N_{|T|}(r),$$

$$N_{T_2}(r) \leqslant 4 N_{|Re(T)|}(r) \leqslant 16 N_{|T|}(r),$$

$$N_{T_3}(r) \leqslant 4 N_{|Im(T)|}(r) \leqslant 16 N_{|T|}(r).$$

By Lemma 5(iv), we get $|\Sigma_{T(\theta)}(r) + \Sigma_{T_2}(r) + \Sigma_{T_3}(r)| \leqslant 72 r N_{|T|}(r)$ for any $r > 0$. Since we have:

$$\left|-\frac{1+e^{i\theta}}{2}\right| = \left|\cos\left(\frac{\theta}{2}\right)\right| \leqslant 1 \quad \text{and} \quad \left|-i\frac{1-e^{i\theta}}{2}\right| = \left|\sin\left(\frac{\theta}{2}\right)\right| \leqslant 1,$$

we get by Lemma 5(ii):

$$\left|\Sigma_{T_2}(r) + \frac{1+e^{i\theta}}{2} \Sigma_{Re(T)}(r)\right| \leqslant r N_{Re(T)}(r) \leqslant 4 r N_{|T|}(r)$$

and

$$\left|\Sigma_{T_3}(r) + i\frac{1-e^{i\theta}}{2} \Sigma_{Im(T)}(r)\right| \leqslant r N_{Im(T)}(r) \leqslant 4 r N_{|T|}(r).$$

It follows that

$$\left|\Sigma_{T(\theta)} - \frac{1+e^{i\theta}}{2} \Sigma_{Re(T)}(r) - i\frac{1-e^{i\theta}}{2} \Sigma_{Im(T)}(r)\right| \leqslant 80 r N_{|T|}(r) \quad \text{for any } r > 0.$$

Integrating this relation with respect to θ and taking the real parts, we get

$$\left|Re\left(\int_0^{2\pi} \Sigma_{T(\theta)}(r) \frac{d\theta}{2\pi}\right) - \frac{1}{2} \Sigma_{Re(T)}(r)\right| \leqslant 80 r N_{|T|}(r),$$

and hence:

$$\Sigma_{Re(T)}(r) \leqslant 2\, Re\left(\int_0^{2\pi} \Sigma_{T(\theta)}(r)\, \frac{d\theta}{2\pi}\right) + 160 r N_{|T|}(r).$$

Since $N_{T(\theta)}(r) = N_{|T(\theta)|}(r) \leqslant 4 N_{|T|}(r)$, we have

$$|\Sigma_{T(\theta)}(r) - \Sigma^{reg}_{T(\theta)}(r)| \leqslant 4er N_{|T|}(r) \leqslant 12 r N_{|T|}(r)$$

by Lemma 7(i), so that finally

$$\Sigma_{Re(T)}(r) \leqslant 2\, Re\left(\int_0^{2\pi} \Sigma^{reg}_{T(\theta)}(r)\, \frac{d\theta}{2\pi}\right) + 184 r N_{|T|}(r).$$

This achieves step 1.

Step 2: *There exists a constant* $K \geqslant 0$ *such that we have, for any* $r > 0$:

$$\Sigma_{Re(T)}(r) \leqslant 2r\left(\int_0^{2\pi} I_r(T(\theta))\, \frac{d\theta}{2\pi}\right) + 184 r N_{|T|}(r) + Kr \ln\left(\frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}}\right),$$

where $I_r(X) = \int_{\sigma^*(T)} h(r^{-1}\lambda)\, d\nu_X(\lambda)$ $(X \in L^p(M,\tau) \cap M)$ *and* $h : \mathbb{C} \to \mathbb{R}$ *is the superharmonic function given by Lemma* 7(ii).

By Lemma 7(ii), there exists a constant $A \geqslant 0$ such that

$$|Re\, \Sigma^{reg}_{T(\theta)}(r) - r I_r(T(\theta))| \leqslant Ar \ln\left(\frac{\Pi_{|T(\theta)|}(r)}{r^{N_{|T(\theta)|}(r)}}\right) \quad \text{for any } r > 0.$$

By Lemma 6(i) and [13, Remark 3.3, p. 280], we have

$$\ln\left(\frac{\Pi_{|T(\theta)|}(r)}{r^{N_{|T(\theta)|}(r)}}\right) = \ln(\Pi_{r^{-1}|T(\theta)|}(1)) = \int_{|\lambda| \geqslant 1} \ln(|\lambda|)\, d\nu_{r^{-1}|T(\theta)|}(\lambda)$$
$$= \int_{\mu_t(r^{-1}T(\theta)) \geqslant 1} \ln(\mu_t(r^{-1}T(\theta)))\, dt.$$

Since $\mu_t(r^{-1}T(\theta)) \leqslant \mu_{t/2}(r^{-1}T)$ for any $t > 0$ (cf. [13, Lemma 2.5(v), p. 276]), we get

$$\ln\left(\frac{\Pi_{|T(\theta)|}(r)}{r^{N_{|T(\theta)|}(r)}}\right) \leqslant \int_{\mu_{t/2}(r^{-1}T) \geqslant 1} \ln(\mu_{t/2}(r^{-1}T))\, dt = 2\int_{\mu_s(r^{-1}T) \geqslant 1} \ln(\mu_s(r^{-1}T))\, ds$$
$$= 2\int_{|\lambda| \geqslant 1} \ln(|\lambda|)\, d\nu_{r^{-1}|T|}(\lambda) = 2\ln(\Pi_{r^{-1}|T|}(1)) = 2\ln\left(\frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}}\right),$$

and hence

$$|Re\, \Sigma^{reg}_{T(\theta)}(r) - r I_r(T(\theta))| \leqslant 2Ar \ln\left(\frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}}\right).$$

Integrating this relation with respect to θ, we get

$$\left|2\left(\int_0^{2\pi} Re\,\Sigma^{reg}_{T(\theta)}(r)\,\frac{d\theta}{2\pi}\right) - 2r\left(\int_0^{2\pi} I_r(T(\theta))\,\frac{d\theta}{2\pi}\right)\right| \leqslant Kr\ln\left(\frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}}\right),$$

where $K = 4A$. From step 1, we deduce that:

$$\Sigma_{Re(T)}(r) \leqslant 2r\left(\int_0^{2\pi} I_r(T(\theta))\,\frac{d\theta}{2\pi}\right) + 184rN_{|T|}(r) + Kr\ln\left(\frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}}\right),$$

and step 2 is achieved.

End of the proof: By Proposition 3, the function

$$z \to 2rI_r\left(\frac{1}{2}(T + zT^*)\right) = 2r\int_{\sigma^*(2^{-1}(T+zT^*))} h(r^{-1}\lambda)\,d\nu_{2^{-1}(T+zT^*)}(\lambda)$$

is superharmonic, and hence $2r\int_0^{2\pi} I_r(T(\theta))\frac{d\theta}{2\pi} \leqslant 2rI_r(\frac{T}{2})$. By Lemma 7(ii) and (i), we deduce that

$$\begin{aligned} 2r\int_0^{2\pi} I_r(T(\theta))\,\frac{d\theta}{2\pi} &\leqslant 2\,Re\,\Sigma^{reg}_{T/2}(r) + 2Ar\ln\left(\frac{\Pi_{|T/2|}(r)}{r^{N_{|T/2|}(r)}}\right) \\ &\leqslant 2\,Re\,\Sigma_{T/2}(r) + 2erN_{T/2} + Kr\ln\left(\frac{\Pi_{|T/2|}(r)}{r^{N_{|T/2|}(r)}}\right). \end{aligned}$$

But we have $|2\,Re\,\Sigma_{T/2}(r) - Re\,\Sigma_T(r)| \leqslant 2rN_T(r)$ by Lemma 5(ii), so that we get by Step 2:

$$\begin{aligned} \Sigma_{Re(T)}(r) \leqslant{}& Re\Sigma_T(r) + 2rN_T(r) + 6rN_{T/2}(r) + Kr\ln\left(\frac{\Pi_{|T/2|}(r)}{r^{N_{|T/2|}(r)}}\right) + 184rN_{|T|}(r) \\ &+ Kr\ln\left(\frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}}\right). \end{aligned}$$

By Lemmas 4(i) and 6(iii)–(ii), we have

$$N_{T/2}(r) = N_T(2r) \leqslant N_T(r) \leqslant \frac{1}{\ln(2)}\ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right),$$

$$N_{|T|}(r) \leqslant \frac{1}{\ln(2)}\ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right),$$

$$\ln\left(\frac{\Pi_{|T|}(r)}{r^{N_{|T|}(r)}}\right) \leqslant \ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right),$$

so that we finally get the existence of a constant $C \geqslant 0$ such that

$$\Sigma_{Re(T)}(r) \leqslant Re\,\Sigma_T(r) + Cr \ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right).$$

By changing T into $-T$, we get $|Re\,\Sigma_T(r) - \Sigma_{Re(T)}(r)| \leqslant Cr \ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right)$, and by changing T into iT in this last inequality, we get $|Im\,\Sigma_T(r) - \Sigma_{Im(T)}(r)| \leqslant Cr \ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right)$. The proof is thus complete. □

4. Spectral characterization of sums of commutators: the general case

4.1. The main result

4.1.1. Extending the definition of σ_T

Let M be a II_∞-factor acting on a separable Hilbert space H and denote by τ its normal faithful semi-finite trace. Let $T \in L^p(M, \tau) \cap M$ $(0 < p < +\infty)$ and consider the Brown's spectral measure ν_T on $\sigma^*(T)$. For any $a > 0$, set $E(a) = \{\lambda \in \sigma^*(T) \mid |\lambda| \geqslant a\}$. By Brown [3, Corollary 3.8(i)], we have $a^p \nu_T(E(a)) \leqslant \int_{E(a)} |\lambda|^p \, d\nu_T(\lambda) \leqslant \int_{\sigma^*(T)} |\lambda|^p \, d\nu_T(\lambda) \leqslant ||T||_p^p < +\infty$, and hence $\nu_T(E(a)) < +\infty$. It follows that the function $\lambda \to |\lambda|$ on $\sigma^*(T)$ admits a non-increasing rearrangement with respect to the Brown's spectral measure ν_T. Let us denote by $t \to \lambda_t(T)$ this non-increasing rearrangement; it is a bounded non-increasing right continuous function that belongs to $L^p([0, +\infty[, dt)$, since we have by Brown [3, Theorem 3.6]:

$$\int_0^{+\infty} \lambda_t(T)^p \, dt \leqslant \int_0^{+\infty} \mu_t(T)^p \, dt = ||T||_p^p < +\infty.$$

In fact, we have a little more:

Lemma 8. *Let X be a rearrangement invariant Köthe function space on $[0, +\infty[$ that is contained in $L^p([0, +\infty[, dt)$ for some $p > 0$. Let M be a II_∞-factor acting on a separable Hilbert space H and denote by τ its normal faithful semi-finite trace. Then, for any element $T \in I_X(M, \tau)$, the function $t \to \lambda_t(T)$ belongs to X.*

Proof. Let us fix $t > 0$. For any $\alpha > 0$, we have by Brown [3, Theorem 3.6]:

$$\lambda_t(T)^\alpha \leqslant \int_0^t \lambda_s(T)^\alpha \frac{ds}{t} \leqslant \int_0^t \mu_s(T)^\alpha \frac{ds}{t}.$$

Taking the power $\frac{1}{\alpha}$ and letting $\alpha \to 0^+$, we get

$$\lambda_t(T) \leqslant \lim_{\alpha \to 0^+} \left(\int_0^t \mu_s(T)^\alpha \frac{ds}{t}\right)^{\frac{1}{\alpha}} = \exp\left(\frac{1}{t}\int_0^t \ln(\mu_s(T)) \, ds\right) = \Lambda_t(T)^{\frac{1}{t}},$$

and since the function $t \to \Lambda_t(T)^{\frac{1}{t}}$ belongs to X by Proposition 2, we get the result. $\square$

Since we have, for any $T \in L^p(M, \tau) \cap M$ $(0 < p < +\infty)$:

$$\int_{\sigma^*(T)} |\lambda| \, d\nu_T(\lambda) = \int_0^{+\infty} \lambda_t(T) \, dt \quad \text{and}$$
$$\int_{|\lambda| > a} |\lambda| \, d\nu_T(\lambda) = \int_{\lambda_t(T) > a} \lambda_t(T) \, dt \quad \text{for any } a > 0,$$

the following definition makes sense:

Definition 2. Let M be an infinite semi-finite von Neumann algebra with a normal faithful semi-finite trace τ. For any $T \in L^p(M, \tau) \cap M$ and any $t > 0$, let $\sigma_T(t)$ be the complex number defined as follows:

$$\sigma_T(t) = \frac{1}{t} \int_{|\lambda| \geqslant \lambda_t(T)} \lambda \, d\nu_T(\lambda) = \frac{1}{t} \int_{|\lambda| > \lambda_t(T)} \lambda \, d\nu_T(\lambda)$$

if $\lambda_t(T) > 0$ and t is not contained in an interval of constancy of $s \to \lambda_s(T)$;

$$\sigma_T(t) = \frac{1}{t} \left(\int_{|\lambda| > \lambda_t(T)} \lambda \, d\nu_T(\lambda) + \frac{t-a}{b-a} \int_{|\lambda| = \lambda_t(T)} \lambda \, d\nu_T(\lambda) \right)$$

if $\lambda_t(T) > 0$ and t belongs to an interval of constancy $[a, b[$ or $[a, b]$ $(a < b)$ of $s \to \lambda_s(T)$, and

$$\sigma_T(t) = \frac{1}{t} \int_{\sigma^*(T)} \lambda \, d\nu_T(\lambda) \quad \text{if } \lambda_t(T) = 0.$$

Remark. If $T = T^* \in L^p(M, \tau) \cap M$, this definition of σ_T coincides by Brown [3, Remark 4.2] with the one given in Definition 1. Note also that we have, as in the self-adjoint case:

$$\left| \sigma_T(t) - \frac{1}{t} \int_{|\lambda| > \lambda_t(T)} \lambda \, d\nu_T(\lambda) \right| \leqslant \lambda_t(T),$$

where (Lemma 8) the function $t \to \lambda_t(T)$ belongs to X if $T \in I_X(M, \tau)$.

4.1.2. Statement of the main result

The following result extends Theorem 1:

Theorem 3. *Let X be a rearrangement invariant Köthe function space on $[0,+\infty[$ that is contained in $L^p([0,+\infty[,dt)$ for some $p>0$. Let M be a II_∞-factor acting on a separable Hilbert space H and denote by τ its normal faithful semi-finite trace. For any element T in $I_X(M,\tau)$, the following conditions are equivalent*:

(i) *T is a finite sum of commutators of the form $[A,B]=AB-BA$ with $A\in I_X(M,\tau)$ and $B\in M$*;

(ii) *the measurable function $t\in]0,+\infty[\to\frac{1}{t}\int_{|\lambda|>\lambda_t(T)}\lambda\,d\nu_T(\lambda)\in\mathbb{C}$ belongs to X*;

(iii) *the measurable function $t\in]0,+\infty[\to\sigma_T(t)\in\mathbb{C}$ belongs to X*;

(iv) *there exists $T_0\in I_X(M,\tau)$ such that $|\sigma_T(t)|\leqslant\mu_t(T_0)$ for almost every $t>0$*;

(v) *there exists $T_0\in I_X(M,\tau)$, $T_0\geqslant 0$, such that $|\Sigma_T(r)|\leqslant rN_{T_0}(r)$ for any $r>0$*;

(vi) *there exists $T_0\in I_X(M,\tau)$, $T_0\geqslant 0$, such that $|\Sigma_T(r)|\leqslant r\ln\left(\frac{\Pi_{T_0}(r)}{r^{N_{T_0}(r)}}\right)$ for any $r>0$.*

Moreover, if one of these conditions is satisfied, T is the sum of less than 14 *commutators of the form $[A,B]=AB-BA$ with $A\in I_X(M,\tau)$ and $B\in M$.*

For a self-adjoint T, equivalence (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) is the content of Theorem 1. Obviously, $T\in I_X(M,\tau)$ satisfies (i) if and only if $Re(T)$ and $Im(T)$ satisfy this condition. However, it is not clear that $T\in I_X(M,\tau)$ satisfies (ii) if and only if both $Re(T)$ and $Im(T)$ satisfy this condition so that we cannot use directly Theorem 1 to prove (i)$\Leftrightarrow$(ii) by reduction to the self-adjoint case. The interest of conditions (iv), (v) and (vi), that are reformulations of (ii), is to show that $T\in I_X(M,\tau)$ satisfies (ii) if and only if $Re(T)$ and $Im(T)$ satisfy this condition. Before proving Theorem 3, let us collect some technical lemmas.

4.1.3. Some technical lemmas

Lemma 9. *Let M be a II_∞-factor acting on a separable Hilbert space H and denote by τ its normal faithful semi-finite trace. For any positive bounded measurable function f on $[0,+\infty[$, there exists $T\in M_+$ such that $\mu_t(T)=f^*(t)$ for any $t>0$, where f^* denotes the non-increasing rearrangement of f.*

Proof. By Proposition 1, there exists an increasing strongly continuous function $\lambda\to E_\lambda$ from $[0,+\infty[$ into the lattice of orthogonal projections in M such that $E_0=0$, $E_\lambda\to I$ when $\lambda\to+\infty$ and $\tau(E_\lambda)=\lambda$ for any $\lambda\geqslant 0$. Set $T=\int_0^{+\infty}f^*(\lambda)\,dE_\lambda$; we thus define a positive element in M which satisfies, for any $a>0$:

$$\begin{aligned}\tau(\mathbb{1}_{]a,+\infty[}(T)) &= \tau\left(\int_{\{\lambda\geqslant 0\,|\,f^*(\lambda)>a\}}dE_\lambda\right)=\int_{\{t\geqslant 0\,|\,f^*(t)>a\}}dt\\ &= mes(\{t\geqslant 0\,|\,f^*(t)>a\}).\end{aligned}$$

By Fack and Kosaki [13, Proposition 2.2, p. 274], we deduce that $\mu_t(T)=f^*(t)$ for any $t>0$. $\square$

Lemma 10. *Let X be a rearrangement invariant Köthe function space on $[0,+\infty[$ that is contained in $L^p([0,+\infty[,dt)$ for some $p>0$. Let M be a II_∞-factor acting on a separable Hilbert space H and denote by τ its normal faithful semi-finite trace. For any $T_1, T_2, \ldots, T_n$ in $I_X(M,\tau)$ and $C_1, C_2, \ldots, C_n$ in $\mathbb{R}_+$, there exists $T \in I_X(M,\tau)$ such that $\prod_{i=1}^n \left(\frac{\Pi_{T_i}(r)}{r^{N_{T_i}(r)}}\right)^{C_i} \leqslant \frac{\Pi_T(r)}{r^{N_T(r)}}$ for any $r>0$.*

Proof. Since we have $\frac{\Pi_S(r)}{r^{N_S(r)}} = \Pi_{r^{-1}S}(1)$ for any $S \in I_X(M,\tau)$ and $r>0$ (cf. Lemma 6(i)), we may assume w.l.o.g. that $r=1$. On the other hand, since we have $\Pi_S(1)^C \leqslant \Pi_S(1)^{[C]+1}$ for any $C \geqslant 0$ and $S \in I_X(M,\tau)$ (because $\Pi_S(1) \geqslant 1$), it suffices to prove the existence, for any T_1, T_2 in $I_X(M,\tau)$, of an element $T \in I_X(M,\tau)$ such that $\Pi_{T_1}(1)\Pi_{T_2}(1) \leqslant \Pi_T(1)$. Since M is a II_∞-factor, there exists by Dixmier [6, Corollaire 3, p. 219] two partial isometries U_1 and U_2 in M with $U_1U_1^* = I, U_2U_2^* = I$, such that the projections $E_1 = U_1^*U_1,\ E_2 = U_2^*U_2$ are orthogonal with sum I. Set $T = S_1 + S_2$, where $S_i = U_i^*T_iU_i\ (i=1,2)$. We thus define an element $T \in I_X(M,\tau)$ that writes $T = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$ in the orthogonal decomposition $H = E_1(H) \oplus E_2(H)$. By Brown [3, Theorems 4.3 and 4.6], we get $\nu_T = \nu_{S_1} + \nu_{S_2} = \nu_{T_1} + \nu_{T_2}$, and hence:

$$\begin{aligned}\Pi_{T_1}(1)\Pi_{T_2}(1) &= \exp\left(\int_{|\lambda|\geqslant 1} \ln(|\lambda|)\, d(\nu_{T_1}+\nu_{T_2})(\lambda)\right)\\ &= \exp\left(\int_{|\lambda|\geqslant 1} \ln(|\lambda|)\, d\nu_T(\lambda)\right) = \Pi_T(1). \qquad \square\end{aligned}$$

4.1.4. Proof of Theorem 3

Let us denote by $C_X(M,\tau)$ the linear span of commutators of the form $[A,B] = AB - BA$ with $A \in I_X(M,\tau)$ and $B \in M$. We shall first prove that we have (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (iv), so that properties (ii)–(vi) will be equivalent. Then, by using Theorems 1 and 2, we shall prove that (i) $\Leftrightarrow$ (vi).

(ii) $\Rightarrow$ (iv): Let $T \in I_X(M,\tau)$ such that $\sigma_T \in X$. We claim that we have, for any $t>0$:

$$|\sigma_T(s)| \leqslant |\sigma_T(t)| + \lambda_t(T) \quad \text{for any } s \geqslant t. \tag{4.1}$$

Case 1: $\lambda_s(T) > 0$ *and s is not contained in an interval of constancy of* $x \to \lambda_x(T)$.

In this case, we have

$$\begin{aligned}|\sigma_T(s)| &\leqslant \frac{1}{s}\left|\int_{|\lambda|\geqslant \lambda_t(T)} \lambda\, d\nu_T(\lambda)\right| + \frac{\lambda_t(T)}{s}\int_{|\lambda|\geqslant \lambda_s(T)} d\nu_T(\lambda)\\ &\leqslant \frac{1}{t}\left|\int_{|\lambda|\geqslant \lambda_t(T)} \lambda\, d\nu_T(\lambda)\right| + \lambda_t(T) = |\sigma_T(t)| + \lambda_t(T).\end{aligned}$$

Case 2: *$\lambda_s(T)>0$ and s belongs to an interval of constancy $[a,b[$ or $[a,b]$ $(a<b)$ of $x\to\lambda_x(T)$.*

In this case, we have

$$\begin{aligned}|\sigma_T(s)| &= \frac{1}{s}\left|\int_{|\lambda|>\lambda_s(T)} \lambda\, d\nu_T(\lambda) + \frac{s-a}{b-a}\int_{|\lambda|=\lambda_s(T)} \lambda\, d\nu_T(\lambda)\right| \\ &\leqslant \frac{1}{s}\left|\int_{|\lambda|>\lambda_t(T)} \lambda\, d\nu_T(\lambda)\right| + \frac{\lambda_t(T)a}{s} + \frac{s-a}{s}\lambda_t(T) \\ &\leqslant \frac{1}{t}\left|\int_{|\lambda|>\lambda_t(T)} \lambda\, d\nu_T(\lambda)\right| + \lambda_t(T).\end{aligned}$$

Case 3: $\lambda_s(T)=0$. In this case, we have $\nu_T(\sigma^*(T))\leqslant s$ and hence

$$\begin{aligned}|\sigma_T(s)| &= \frac{1}{s}\left|\int_{|\lambda|\geqslant\lambda_t(T)} \lambda\, d\nu_T(\lambda) + \int_{0<|\lambda|\leqslant\lambda_t(T)} \lambda\, d\nu_T(\lambda)\right| \\ &\leqslant \frac{1}{s}\left|\int_{|\lambda|\geqslant\lambda_t(T)} \lambda\, d\nu_T(\lambda)\right| + \frac{\lambda_t(T)}{s}\nu_T(\sigma^*(T)) \leqslant |\sigma_T(t)| + \lambda_t(T).\end{aligned}$$

Formula (4.1) is thus proved. We deduce from (4.1) that the non-increasing function $f(t)=\mathrm{Sup}_{s\geqslant t}|\sigma_T(s)|$ satisfies $f(t)\leqslant|\sigma_T(t)|+\lambda_t(T)\leqslant 2\lambda_t(T)+|\frac{1}{t}\int_{|\lambda|>\lambda_t(T)}\lambda\, d\nu_T(\lambda)|$ for any $t>0$, and hence belongs to X. By Lemma 9, there exists $T_0\in I_X(M,\tau)_+$ such that $f(t+0)=\mu_t(T_0)$ for any $t>0$, and since we have $|\sigma_T(t)|\leqslant f(t)=f^*(t)=\mu_t(T_0)$ for almost every $t>0$, we get (iv).

(iv) $\Rightarrow$ (iii): The condition $|\sigma_T(t)|\leqslant\mu_t(T_0)$ for almost every $t>0$ implies that $|\sigma_T|^*(t)\leqslant\mu_t(T_0)$ for any $t>0$, and hence that $\sigma_T\in X$.

(iii) $\Rightarrow$ (ii): Follows immediately from the relation:

$$\left|\frac{1}{t}\int_{|\lambda|>\lambda_t(T)} \lambda\, d\nu_T(\lambda)\right| \leqslant \lambda_t(T) + |\sigma_T(t)|,$$

since the function $t\to\lambda_t(T)$ belongs to X by Lemma 8.

(iv) $\Rightarrow$ (v): We may assume w.l.o.g. that T_0 is positive. By Lemmas 8 and 9, we may assume in addition that $\lambda_t(T)<\mu_t(T_0)$ and $\mu_t(T_0)>0$ for any $t>0$. Let us show that we have, for any fixed $r>0$:

$$|\Sigma_T(r)|\leqslant 3rN_{T_0}(r). \tag{4.2}$$

Case 1: *$\mu_t(T_0)<r$ for any $t\geqslant 0$.* In this case we have $\lambda_t(T)<r$ for any $t>0$ and hence:

$$|\Sigma_T(r)|\leqslant \int_{\lambda_t(T)\geqslant r} \lambda_t(T)\, dt = 0 \leqslant 3rN_{T_0}(r).$$

Case 2: *There exists* $t \geqslant 0$ *such that* $\mu_t(T_0) \geqslant r$. Let us set in this case $t_0 = Sup\{t \geqslant 0 \mid \mu_t(T_0) \geqslant r\}$. Since the function $t \to \mu_t(T_0)$ is right continuous, we have $\mu_{t_0}(T_0) \leqslant r$ and hence $\lambda_{t_0}(T) < r$. It follows that

$$\begin{aligned}|\Sigma_T(r)| = \left|\int_{|\lambda| \geqslant r} \lambda \, d\nu_T(\lambda)\right| &\leqslant \left|\int_{\lambda_{t_0}(T) < |\lambda|} \lambda \, d\nu_T(\lambda)\right| + \left|\int_{\lambda_{t_0}(T) < |\lambda| < r} \lambda \, d\nu_T(\lambda)\right| \\ &\leqslant t_0(\sigma_T(t_0) + \lambda_{t_0}(T)) + r \int_{\lambda_{t_0}(T) < |\lambda|} d\nu_T(\lambda) \\ &\leqslant 2t_0 \mu_{t_0}(T_0) + rt_0 \leqslant 3rt_0.\end{aligned}$$

But $N_{T_0}(r) = |\{t \geqslant 0 \mid \mu_t(T_0) \geqslant r\}| = t_0$, so that we get $|\Sigma_T(r)| \leqslant 3rN_{T_0}(r)$. This proves (4.2).

Since M is a II_∞-factor, there exists by Dixmier [6, Corollaire 3, p. 219] three partial isometries U_1, U_2 and U_3 in M with $U_i U_i^* = I$, such that the projections $E_i = U_i^* U_i$ $(i = 1, 2, 3)$ are pairwise orthogonal with sum I. Set $S_0 = \sum_{i=1}^3 U_i^* T_0 U_i$ $(i = 1, 2)$. We thus define a positive element $S_0 \in I_X(M, \tau)$ that can be written $S_0 = \begin{pmatrix} U_1^* T_0 U_1 & & 0 \\ & U_2^* T_0 U_2 & \\ 0 & & U_3^* T_0 U_3 \end{pmatrix}$ in the orthogonal decomposition $H = E_1(H) \oplus E_2(H) \oplus E_3(H)$. We deduce that $N_{S_0}(r) = 3N_{T_0}(r)$, and (v) follows from (4.2).

(v) $\Rightarrow$ (vi): Follows from the relation $N_{T_0}(r) \leqslant \ln\left(\frac{\Pi_{eT_0}(r)}{r^{N_{eT_0}(r)}}\right)$ for any $r > 0$ (cf. Lemma 6(iii)).

(vi) $\Rightarrow$ (iv): Replacing T_0 by $|T| + T_0$ if necessary, we may assume that $\mu_t(T) \leqslant \mu_t(T_0)$ for any $t \geqslant 0$ since the function

$$S \to \ln\left(\frac{\Pi_S(r)}{r^{N_S(r)}}\right) = \ln(\Pi_{r^{-1}S}(1)) = \int_{|\lambda| \geqslant 1} \ln_+(|\lambda|) d\nu_{r^{-1}S}(\lambda) = \int_0^{+\infty} \ln_+(r^{-1}\mu_t(S)) \, dt$$

is non-decreasing on $M_+ \cap L^p(M, \tau)$. By Lemmas 8, 9 and Proposition 2, we may assume in addition that $\lambda_t(T) < \mu_t(T_0)$ and $\mu_t(T_0) > 0$ for any $t > 0$. Let us show that we have, for any $t > 0$:

$$|\sigma_T(t)| \leqslant \frac{1}{t} |\Sigma_T(\mu_t(T_0))| + \mu_t(T_0). \tag{4.3}$$

Case 1: $\lambda_t(T) > 0$ *and* t *is not contained in an interval of constancy of* $s \to \lambda_s(T)$. In this case, we have

$$\begin{aligned}|\sigma_T(t)| &= \frac{1}{t}\left|\int_{|\lambda| \geqslant \lambda_t(T)} \lambda \, d\nu_T(\lambda)\right| \leqslant \frac{1}{t}\left|\int_{|\lambda| \geqslant \mu_t(T_0)} \lambda \, d\nu_T(\lambda)\right| + \frac{\mu_t(T_0)}{t} \int_{|\lambda| \geqslant \lambda_t(T)} d\nu_T(\lambda) \\ &\leqslant \frac{1}{t} |\Sigma_T(\mu_t(T_0))| + \mu_t(T_0).\end{aligned}$$

Case 2: $\lambda_t(T) > 0$ *and t belongs to an interval of constancy* $[a, b[$ *or* $[a, b]$ $(a < b)$ *of* $s \to \lambda_s(T)$. In this case, we have

$$|\sigma_T(t)| = \frac{1}{t}\left| \int_{|\lambda| \geqslant \mu_t(T_0)} \lambda \, d\nu_T(\lambda) + \int_{\lambda_t(T) < |\lambda| < \mu_t(T_0)} \lambda \, d\nu_T(\lambda) + \frac{t-a}{b-a} \int_{|\lambda| = \lambda_t(T)} \lambda \, d\nu_T(\lambda) \right|$$
$$\leqslant \frac{1}{t}|\Sigma_T(\mu_t(T_0))| + \frac{\mu_t(T_0)}{t} \int_{\lambda_t(T) < |\lambda|} d\nu_T(\lambda) + \frac{\lambda_t(T)}{t}\left(\frac{t-a}{b-a}\right) \int_{\lambda_t(T) = |\lambda|} d\nu_T(\lambda)$$
$$\leqslant \frac{1}{t}|\Sigma_T(\mu_t(T_0))| + \frac{1}{t}[\mu_t(T_0)a + \mu_t(T_0)(t-a)] = \frac{1}{t}|\Sigma_T(\mu_t(T_0))| + \mu_t(T_0).$$

Case 3: $\lambda_t(T) = 0$. In this case, we have $\nu_T(\sigma^*(T)) \leqslant t$ and hence:

$$|\sigma_T(t)| = \frac{1}{t}\left| \int_{|\lambda| \geqslant \mu_t(T_0)} \lambda \, d\nu_T(\lambda) + \int_{0 < |\lambda| \leqslant \mu_t(T_0)} \lambda \, d\nu_T(\lambda) \right|$$
$$\leqslant \frac{1}{t}|\Sigma_T(\mu_t(T_0))| + \frac{\mu_t(T_0)}{t} \nu_T(\sigma^*(T)) \leqslant \tfrac{1}{t}|\Sigma_T(\mu_t(T_0))| + \mu_t(T_0).$$

By Lemma 6(i) and the hypothesis, we get

$$\frac{1}{t}|\Sigma_T(\mu_t(T_0))| \leqslant \frac{1}{t}\mu_t(T_0) \ln(\Pi_{\mu_t(T_0)^{-1}T_0}(1)) = \frac{\mu_t(T_0)}{t} \int_0^{+\infty} \ln_+(\mu_t(T_0)^{-1}\mu_s(T_0)) \, ds$$
$$= \frac{\mu_t(T_0)}{t} \int_0^t \ln(\mu_t(T_0)^{-1}\mu_s(T_0)) \, ds.$$

Since we have $x \leqslant e^x$ for any $x \geqslant 0$, we get

$$\frac{1}{t}|\Sigma_T(\mu_t(T_0))| \leqslant \mu_t(T_0) \exp\left(\frac{1}{t}\int_0^t \ln(\mu_t(T_0)^{-1}\mu_s(T_0)) \, ds\right) = \exp\left(\frac{1}{t}\int_0^t \ln(\mu_s(T_0)) \, ds\right),$$

and hence by (4.3)

$$|\sigma_T(t)| \leqslant \exp\left(\frac{1}{t}\int_0^t \ln(\mu_s(T_0)) \, ds\right) + \mu_t(T_0) \quad \text{for any } t > 0.$$

Since the function $t \to \exp(\frac{1}{t}\int_0^t \ln(\mu_s(T_0)) \, ds)$ is non-increasing and belongs to X by Proposition 2, there exists by Lemma 9 an element $S_0 \in I_X(M, \tau)$ such that

$$\exp\left(\frac{1}{t}\int_0^t \ln(\mu_s(T_0)) \, ds\right) + \mu_t(T_0) = \mu_t(S_0) \quad \text{for any } t > 0,$$

and condition (iv) is proved.

(i) $\Rightarrow$ (vi): Let $T \in C_X(M, \tau)$. Since $T_1 = Re(T)$ and $T_2 = Im(T)$ are in $C_X(M, \tau)$, they satisfy condition (ii) by Theorem 1. But we know that (ii) $\Rightarrow$ (vi) so that there exist two positive operators $S_1, S_2 \in I_X(M, \tau)$, such that $|\Sigma_{T_i}(r)| \leqslant r \ln\left(\frac{\Pi_{S_i}(r)}{r^{N_{S_i}(r)}}\right)$ for any

$r>0$ and $i=1,2$. By Theorem 2, there exists a constant $C>0$ such that

$$\begin{aligned}|\Sigma_T(r)| &\leqslant 2Cr\ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right)+|\Sigma_{T_1}(r)|+|\Sigma_{T_2}(r)|\\ &\leqslant 2Cr\ln\left(\frac{\Pi_{2|T|}(r)}{r^{N_{2|T|}(r)}}\right)+r\ln\left(\frac{\Pi_{S_1}(r)}{r^{N_{S_1}(r)}}\right)+r\ln\left(\frac{\Pi_{S_2}(r)}{r^{N_{S_2}(r)}}\right),\end{aligned}$$

and (vi) follows from Lemma 10.

(vi) $\Rightarrow$ (i): Assume that $T\in I_X(M,\tau)$ satisfies (vi). By Theorem 2 and Lemma 10, the operators $T_1=Re(T)$ and $T_2=Im(T)$ also satisfy (vi). But we know that (vi) $\Rightarrow$ (ii) so that the self-adjoint operators $T_1=Re(T)$ and $T_2=Im(T)$ satisfy (ii) and hence (i) by Theorem 1. This implies that $T=T_1+iT_2\in C_X(M,\tau)$. $\square$

4.2. Application to Dixmier's traces

4.2.1. Dixmier's traces on Marcinkiewicz spaces

Let M be a II_∞ factor and denote by τ its normal faithful semi-finite trace. For any Lorentz weight ρ, consider the Marcinkiewicz space:

$$M^1_\rho(M,\tau)=\left\{T\in\widetilde{M}\mid \exists C>0 \text{ such that } \int_0^t \mu_s(T)\,ds\leqslant C\int_0^t\rho(s)\,ds\right\}$$

with the norm $||T||^*_\rho=\mathrm{Sup}_{t>0}\left(\frac{\int_0^t\mu_s(T)\,ds}{\int_0^t\rho(s)\,ds}\right)$. This non-commutative Banach space identifies with the dual of $L^1_\rho(M,\tau)$ (cf. [21, Proposition 1.8, p. 22]). We shall assume here that ρ is bounded and that $(\frac{1}{t}\int_0^t\rho(s)\,ds)^p$ is integrable on $[1,+\infty[$ for some $p>0$, so that $M^1_\rho(M,\tau)\subset M\cap L^p(M,\tau)$. For $\rho(t)=\frac{1}{1+t}$, we recover the Dixmier's ideal:

$$L^{1,\infty}(M,\tau)=\left\{T\in M\mid \exists C>0 \text{ such that } \int_0^t\mu_s(T)\,ds\leqslant C\ln(1+t)\right\}$$

which obviously contains $M\cap L^1(M,\tau)$ and is contained in $M\cap L^p(M,\tau)$ for any $p>1$. The Dixmier's ideal appears naturally in the index theory of elliptic operators affiliated with von Neumann algebras such as uniformly elliptic almost periodic pseudo-differential operators on $\mathbb{R}^n$ (cf. [22]) or leafwise elliptic pseudo-differential operators on measured foliations [2]. To define a Dixmier's trace on $M^1_\rho(M,\tau)$, we need an *averaging procedure* LIM_ω i.e. a positive linear form $f\to LIM_\omega(f(t))$ on $L^\infty_R([0,+\infty[)$ satisfying

(i) $\liminf \mathrm{ess}_{t\to+\infty} f(t)\leqslant LIM_\omega(f(t))\leqslant \limsup \mathrm{ess}_{t\to+\infty} f(t)$ for any $f\in L^\infty_R([0,+\infty[)$;
(ii) $LIM_\omega(f(\lambda t))=LIM_\omega(f(t))$ for any $f\in L^\infty_R([0,+\infty[)$ and any $\lambda>0$.

It is easy to show the existence of such averaging procedures (see for instance [21, p. 27]). If the Lorentz weight ρ satisfies $\lim_{t\to+\infty}\left(\frac{\int_0^t \rho(s)\,ds}{\int_0^{2t}\rho(s)\,ds}\right)=1$, we can associate to any *averaging procedure* LIM_ω a (non-normal) trace τ_ω on $M^1_\rho(M,\tau)$ by setting

$$\tau_\omega(T) = LIM_\omega\left(\frac{1}{\int_0^t \rho(s)\,ds}\int_0^t \mu_s(T)\,ds\right) \text{ if } T \text{ is a positive element in } M^1_\rho(M,\tau).$$

For a general $T\in M^1_\rho(M,\tau)$, set

$$\tau_\omega(T) = \tau_\omega(Re(T)_+) - \tau_\omega(Re(T)_-) + i\tau_\omega(Im(T)_+) - i\tau_\omega(Im(T)_-).$$

We thus get a continuous positive linear form $\tau_\omega : M^1_\rho(M,\tau)\to\mathbb{C}$ satisfying

$$\tau_\omega(TS) = \tau_\omega(ST) \quad \text{for any } T\in M^1_\rho(M,\tau) \text{ and } S\in M.$$

Let us call such a linear form τ_ω a *Dixmier's trace* on $M^1_\rho(M,\tau)$. The Dixmier's traces appear naturally in non-commutative geometry. For instance, we have:

Proposition (Benameur and Fack [2]). *Let (M,F,Λ) be a measured p-dimensional foliation on a smooth compact manifold M without boundary. For any riemannian metric on M and any smooth hermitian complex vector bundle E on M, denote by $\tau=\tau_{\Lambda,E}$ the normal semi-finite trace on the von Neumann algebra $N=W^*(M,F,\Lambda,End(E))$ associated with (M,F,Λ) and E. Let P be a leafwise pseudodifferential operator of order $-p$ acting on the sections of E, with principal symbol $\sigma_{-p}(P)(x,\xi)$ of order $-p$. Then, we have*:

(i) *P defines an element in the Dixmier's ideal $L^{1,\infty}(N,\tau)$;*

(ii) *the Dixmier's trace of P is given by $\tau_\omega(P)=\int_{M/F} Re\,s_L(P)\,d\Lambda(L)$, where $Re\,s_L(P)$ is the one-density on the leaf manifold defined by*

$$Re\,s_L(P)_x = \frac{dvol_L(x)}{p(2\pi)^p}\int_{|\xi|=1} tr_E(\sigma_{-p}(P)(x,\xi))\,d|\xi|.$$

The Dixmier's trace τ_ω is thus strongly related to the foliated Wodzicki residue.

4.2.2. Spectrality of the Dixmier's traces

The Dixmier's trace $\tau_\omega(T)$ of a positive $T\in M^1_\rho(M,\tau)$ is obviously determined by the τ-spectral measure of T. The following proposition extends this result to non-self-adjoint elements in $M^1_\rho(M,\tau)$:

Proposition 4. *Let M be a II_∞ factor, ρ a bounded Lorentz weight on $]0,+\infty[$ such that $(\frac{1}{t}\int_0^t \rho(s)\,ds)^p$ is integrable on $[1,+\infty[$ for some $p>0$. Assume that*

$\lim_{t\to+\infty}\left(\frac{\int_0^t \rho(s)\,ds}{\int_0^{2t}\rho(s)\,ds}\right) = 1$ *and denote by* τ_ω *a Dixmier's trace on* $M_\rho^1(M,\tau)$. *For any* $T\in M_\rho^1(M,\tau)$ *such that* 0 *is isolated in* $\sigma(T)$, *we have* $\tau_\omega(T)=0$. *In particular,* $\tau_\omega(T)=0$ *for any element* $T\in M_\rho^1(M,\tau)$ *which is quasi-nilpotent or with finite spectrum.*

Proof. Let $T\in M_\rho^1(M,\tau)$ such that 0 is isolated in $\sigma(T)$. Let $D=\{z\in\mathbb{C}\mid |z|\leqslant\varepsilon\}$ be a disk such that $\sigma(T)\cap D=\{0\}$, and consider the Riesz idempotent:

$$P_0 = -\frac{1}{2i\pi}\int_{\partial D}(T-zI)^{-1}\,dz\in M.$$

Let $P\in M$ be the self-adjoint projection with same range as P_0. We have $T=N+R$, where $N=PTP\in M_P$ satisfies $\sigma(N)=\{0\}$ and $R=(I-P)T(I-P)\in M_{I-P}$ is invertible in the reduced von Neumann algebra M_{I-P}. Since $T\in M_\rho^1(M,\tau)\subset M\cap L^p(M,\tau)$ is τ-compact, we have $R=(I-P)T(I-P)\in K_{\tau_{I-P}}(M_{I-P})$. But R is invertible in M_{I-P} so that $\tau(I-P)<+\infty$. This implies that $R\in M\cap L^1(M,\tau)$ and hence $\tau_\omega(R)=0$ by Prinzis [21, Proposition 2.2, p. 29]. On the other hand, $\tau(P)=+\infty$ so that M_P is still a II_∞-factor. Since $\sigma(N)=\{0\}$, we have $\Sigma_N(r)=0$ for any $r>0$ and Theorem 3 implies that $N\in M_\rho^1(M_P,\tau_P)$ is a finite sum of commutators of the form $[A,B]=AB-BA$ with $A\in M_\rho^1(M,\tau)$ and $B\in M$. It follows that $\tau_\omega(N)=0$ and hence $\tau_\omega(T)=\tau_\omega(N)+\tau_\omega(R)=0$. $\square$

As a corollary, we get that $\tau_\omega(T)=0$ for any $T\in L^{1,\infty}(M,\tau)$ such that 0 is isolated in $\sigma(T)$. For instance, $\tau_\omega(T)=0$ if $T\in L^{1,\infty}(M,\tau)$ is quasi-nilpotent or has finite spectrum.

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